Semantic Web journal, under review 0 (2012) 1–0
IOS Press

Paraconsistent OWL and Related Logics

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Abstract. The Web Ontology Language OWL is currently the most prominent formalism for representing ontologies in Semantic Web applications. OWL is based on description logics, and automated reasoners are used to infer knowledge implicitly present in OWL ontologies. However, because typical description logics obey the classical principle of explosion, reasoning over inconsistent ontologies is impossible in OWL. This is so despite the fact that inconsistencies are bound to occur in many realistic cases, e.g., when multiple ontologies are merged or when ontologies are created by machine learning or data mining tools.

In this paper, we present four-valued paraconsistent description logics which can reason over inconsistencies. We focus on logics corresponding to OWL DL and its profiles. We present the logic $\mathcal{SROIQ}4$, showing that it is both sound relative to classical $\mathcal{SROIQ}$ and that its embedding into $\mathcal{SROIQ}$ is consequence preserving. We also examine paraconsistent varieties of $\mathcal{EL}^{++}$, DL-Lite, and Horn-DLs. The general framework described here has the distinct advantage of allowing classical reasoners to draw sound but nontrivial conclusions from even inconsistent knowledge bases. Truth-value gaps and gluts can also be selectively eliminated from models (by inserting additional axioms into knowledge bases). If gaps but not gluts are eliminated, additional classical conclusions can be drawn without affecting paraconsistency.

Keywords: Web Ontology Language, OWL, Description Logic, Paraconsistency, Semantic Web, Complexity, Automated Deduction

1. Introduction

The Semantic Web is based fundamentally on the idea that the usefulness of data on the Web can be significantly increased by publishing it together with corresponding *metadata*, the latter giving a formal and machine processable account of the data's *meaning* [31,32]. The Web Ontology Language [30] OWL, now a W3C recommended standard, specifies this metadata using *ontologies*—logical knowledge bases that precisely describe a given domain of interest.

OWL is closely related to description logics (DLs). Specifically, OWL DL corresponds to the description logic $\mathcal{SROIQ}$ [34], while the profiles OWL EL, OWL QL and OWL RL [57] correspond to the DLs $\mathcal{EL}^{++}$ [8], DL-Lite$_{\mathcal{R}}$ [12], and DLP [28], respectively. The formal semantics of OWL, which gives meaning to OWL knowledge bases and determines their logical consequences, is based on DL semantics, and the automated tools used to reason over OWL knowledge bases are typically based on algorithms developed for description logics.

Standard description logics are in turn closely related to classical first-order logic [31]. In particular, they have a set-theoretic semantics that espouses the law of noncontradiction (no statement can be simultaneously true and false), and the DL account of logical entailment is essentially the same as classical logic's: A set K of statements entails a statement $\mathcal{P}$ if and only if there is no interpretation making all of the members of K true and $\mathcal{P}$ false.

A result of this is that the formal semantics of standard description logics is rendered useless in the presence of contradictory information. Since the statements of an inconsistent knowledge base cannot simultaneously be true, *everything* is en-

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tailed by them. This is called the *principle of explosion*.

$$\text{If } K \models \mathcal{P} \text{ and } K \models \neg\mathcal{P}, \text{ then } K \models \mathcal{Q} \text{ for all } \mathcal{Q}.$$

For any statement $\mathcal{P}$ and its negation $\neg\mathcal{P}$, and any set of statements K, if both $\mathcal{P}$ and $\neg\mathcal{P}$ are entailed by K, then all statements $\mathcal{Q}$ are also entailed by K. The principle holds in all standard description logics, and it means that these logics cannot be used to reason over inconsistencies. In Popper's words, "A theory which adds to every information which it asserts also the negation of this information can give us no information at all" [61].

If inconsistencies were rare or could be easily identified, then this problem would be uninteresting. However, real knowledge bases are distributed and multi-authored, or created using inductive methods or by merging other knowledge bases, and it is unreasonable to expect them to be logically consistent all of the time. Furthermore, it is well known that ascertaining consistency in expressive formal systems is quite difficult (sometimes impossible).

And so it is important to develop ways of dealing with inconsistencies. One method of doing this is to maintain the principle of explosion, treating inconsistencies as errors to be repaired (e.g., by rejecting or modifying statements until consistency is restored) [39,66]. This approach has the virtue of allowing classical logic to be used, albeit over an altered set of statements. However, in addition to the difficulty of determining whether or not a knowledge base is inconsistent (and also of pinpointing the sources of inconsistencies), the approach has the drawback of deleting or modifying some subset of statements. This necessarily removes valuable information.

An alternative solution is to reject the principle of explosion itself, employing a so-called *paraconsistent logic*, i.e., a logic in which inconsistencies do not entail everything. Using one, classically sound and informative consequences can be drawn from inconsistent knowledge bases.

The paraconsistency approach is the one taken here. Extending results found in a collection of papers [47,48,49,50,51], we present 4-valued paraconsistent semantics for selected DLs and show how knowledge bases under the 4-valued semantics can be embedded (i.e. translated) into corresponding knowledge bases under classical DL semantics. The virtue of the approach is that the embedding allows existing classical reasoners to draw inferences according to the new semantics (which, again, are classically sound).

We focus on the logic $\mathcal{SROIQ}4$, the paraconsistent variant of $\mathcal{SROIQ}$. We prove that $\mathcal{SROIQ}4$ is sound *wrt* $\mathcal{SROIQ}$ and so can be used to correctly reason over $\mathcal{SROIQ}$ knowledge bases. We also show that the embedding of $\mathcal{SROIQ}4$ into $\mathcal{SROIQ}$ is consequence preserving: a statement is entailed in $\mathcal{SROIQ}4$ if and only if its translation is entailed in $\mathcal{SROIQ}$. Furthermore, we show that by adding axioms of a special type, the law of noncontradiction and the law of the excluded middle can be selectively enforced in $\mathcal{SROIQ}4$ knowledge bases. This allows one to draw further classically correct conclusions while at the same time (and with certain caveats) maintaining paraconsistency.

We also apply the paraconsistent framework to several varieties of tractable description logic, including Horn-DLs [43], logics in the DL-Lite family, $\mathcal{EL}^{++}$, and the tractable rule language ELP [44]. As with $\mathcal{SROIQ}$, the paraconsistent varieties of these can be embedded into their classical counterparts. In order to ensure tractability, however, restrictions must be specified in each case.

Importantly, we also point out a limitation of the paraconsistent framework described here. Specifically, while many DL constructs do not affect paraconsistency one way or another, the interaction of nominals and cardinality restrictions prevents some knowledge bases from having four-valued models. For these, the principle of explosion still applies.

The remainder of the paper is organized as follows: Section 2 gives a brief overview of multivalued logics and of the difficulties (such as the failure of Modus Ponens) encountered in the 4-valued context. The implication operators (material, internal, and strong) forming the basis for concept inclusion in $\mathcal{SROIQ}4$ are also presented. The syntax and semantics for $\mathcal{SROIQ}4$ are defined in Section 3, and $\mathcal{SROIQ}4$ is shown to be sound relative to $\mathcal{SROIQ}$. Section 4 discusses removing truth value gaps and gluts from $\mathcal{SROIQ}4$ by adding further axioms. The embedding of $\mathcal{SROIQ}4$ into $\mathcal{SROIQ}$ is given in Section 5, as are the theorems asserting that the embedding is consequence preserving. Section 6 shows how the interaction of nominals and cardinality restrictions causes paraconsistency to break down. Sections 7 and 8 discuss embedding tractable logics into their classical counterparts. It is shown there that tractability is lost if translations are based on strong or material inclusion. A series of empirical tests on OWL ontologies is described in Section 9. Although the experiments are limited and of a tentative nature, their results reinforce the theoretical results of earlier sections: efficient reasoning performance can be obtained if internal inclusion is used, but typically not if material or strong inclusion is used. Section 10 discusses related work.

Proofs for results not reported elsewhere are included in the Appendix.

Acknowledgements: This work is a longer version of materials appearing in [46] and [53], which are in turn based upon earlier research [47,48,49,50, 51]. The current work was performed while the first coauthor was a postdoctoral fellow at the Florida Institute for Human and Machine Cognition, and later while at Wright State University. The work was partially supported by the National Science Foundation under award 10172225 "III: Small: TROn—Tractable Reasoning with Ontologies," and partly realized as part of the Quaero Programme, funded by OSEO, the French state agency for innovation.

2. Multivalued Paraconsistent Logics

A logic is paraconsistent if the principle of explosion fails in at least one case, that is if there exist statements $\mathcal{P}$ and $\mathcal{Q}$, and a set of statements K, such that $K \models \mathcal{P}$ and $K \models \neg\mathcal{P}$ but $K \not\models \mathcal{Q}$. Classical logic fails to be paraconsistent because no set entailing $\mathcal{P}$ and $\neg\mathcal{P}$ possesses a model. In the 4-valued logics upon which $\mathcal{SROIQ}4$ is based, paraconsistency is achieved essentially by redefining what "interpretation" and "model" mean, so that classically inconsistent sets can have models.

We will briefly discuss multivalued paraconsistent logics here. The important properties of multivalued logics can be illustrated with propositional logics over the usual set of connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (inclusive disjunction), and $\mapsto$ (material implication). As such, we'll restrict ourselves to discussing these.

An *n-valued* logic specifies a set V of truth values, $|V| = n$, and some subset $D \subseteq V$ of *designated* values. An *n-valued interpretation* assigns elements of V to each atomic statement, and the values of compound statements are defined using truth-functions. An *n-valued model* of K is an interpretation where each statement of K is given a value from D, and K entails $\mathcal{P}$ *iff* every n-valued model of K is also an n-valued model of $\mathcal{P}$.

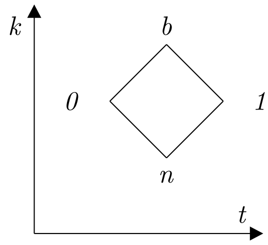

Fig. 1. Belnap's Bilattice

In the context of computing, the use of four truth values $\{n, 0, 1, b\}$ can be traced to Belnap [9,10], with the four values corresponding to what a computer knows or has been told about $\mathcal{P}$: nothing (n); that $\mathcal{P}$ is false (0); that $\mathcal{P}$ is true (1); that $\mathcal{P}$ is true and that $\mathcal{P}$ is false (b). Belnap intended the four values to be treated epistemically. They correspond to what the computer has been told about $\mathcal{P}$, rather than denoting an actual truth value.

The semantics for the logical connectives can be given using Belnap's Bilattice (Figure 1). The truth values are partially ordered according to two relations: $\leq_t$ (the *truth ordering*), and $\leq_k$ (the *knowledge ordering*). Both define complete lattices over the set of truth values. The truth-functions for $\wedge$ and $\vee$ are defined as

- $v(\mathcal{P} \wedge \mathcal{Q}) = glb_{\leq_t}(v(\mathcal{P}), v(\mathcal{Q}))$
- $v(\mathcal{P} \vee \mathcal{Q}) = lub_{\leq_t}(v(\mathcal{P}), v(\mathcal{Q}))$

where $glb_{\leq_t}(x, y)$ and $lub_{\leq_t}(x, y)$ refer, respectively, to the greatest lower bound and least upper bound of x and y in the $\leq_t$-lattice. Negation inverts the diagram in the $\leq_t$-ordering. This semantics yields the characteristic truth tables given in Table 1, where material implication $\mathcal{P} \mapsto \mathcal{Q}$ is defined as $\neg\mathcal{P} \vee \mathcal{Q}$.

The value b is a truth value *glut*, while n is a truth value *gap*. If one allows gluts but not gaps, then one gets Graham Priest's 3-valued *Logic of Paradox* (LP) [62]. Allowing gaps but not gluts yields Kleene's strong 3-valued logic K_3 [42]. If one

Table 1

4-valued semantics for the traditional connectives.

$\neg$	
n	n
0	1
1	0
b	b

$\wedge$	n	0	1	b
n	n	0	n	0
0	0	0	0	0
1	n	0	1	b
b	0	0	b	b

$\vee$	n	0	1	b
n	n	n	1	1
0	n	0	1	b
1	1	1	1	1
b	1	b	1	b

$\mapsto$	n	0	1	b
n	n	n	1	1
0	1	1	1	1
1	n	0	1	b
b	1	b	1	b

allows neither gaps nor gluts, then one is left with classical logic. Typically, if gluts are allowed in a logic L and b is designated, then L is paraconsistent. E.g, LP is paraconsistent, while K_3 is not.

It should be pointed out that, ceteris paribus, all of the consequence relations defined using the multivalued framework possess the following properties.

- **Reflexivity**: If $\mathcal{P} \in K$, then $K \models \mathcal{P}$.
- **Monotonicity**: If $K \models \mathcal{P}$, then for all K', $K \cup K' \models \mathcal{P}$.
- **Transitivity**: If $K \models \mathcal{P}$ for all $\mathcal{P} \in K'$, and $K' \models \mathcal{Q}$, then $K \models \mathcal{Q}$.

The same holds for all of the paraconsistent description logics defined in later sections.

Several traditional rules of inference, however, often do not hold in the 4-valued context. In particular some involving material implication do not hold. Because of this, alternative implication operators (symbolized here using $\leadsto$) have been proposed. Below, $\mathcal{P}$ and $\mathcal{Q}$ are arbitrary statements, and K is an arbitrary set of statements.

- **Disjunctive Syllogism**: If $K \models \neg\mathcal{P}$ and $K \models \mathcal{P} \vee \mathcal{Q}$, then $K \models \mathcal{Q}$.
- **Modus Ponens**: If $K \models \mathcal{P} \leadsto \mathcal{Q}$ and $K \models \mathcal{P}$, then $K \models \mathcal{Q}$.
- **Modus Tollens**: If $K \models \mathcal{P} \leadsto \mathcal{Q}$ and $K \models \neg\mathcal{Q}$, then $K \models \neg\mathcal{P}$.
- **Identity**: $K \models \mathcal{P} \leadsto \mathcal{P}$.
- **Transposition**: If $K \models \mathcal{P} \leadsto \mathcal{Q}$, then $K \models \neg\mathcal{Q} \leadsto \neg\mathcal{P}$.
- **Deduction Theorem**: If $K \cup \{\mathcal{P}\} \models \mathcal{Q}$, then $K \models \mathcal{P} \leadsto \mathcal{Q}$.
- **Strong Equivalence**: For all truth-value assignments v, $[v(\mathcal{P} \leadsto \mathcal{Q}) \in \{1, b\}$ and $v(\mathcal{Q} \leadsto \mathcal{P}) \in \{1, b\}]$ *iff* $v(\mathcal{P}) = v(\mathcal{Q})$.
- **Supraclassicality**: For all truth-value assignments v, if $v(\mathcal{P}) \cup v(\mathcal{Q}) \subseteq \{0, 1\}$, then $v(\mathcal{P} \leadsto \mathcal{Q}) = v(\mathcal{P} \mapsto \mathcal{Q})$.

Let $\models_4$ indicate the consequence relation defined over the propositional language of $\vee$, $\wedge$, $\neg$, and $\mapsto$, and using $V = \{n, 0, 1, b\}$ and $D = \{1, b\}$. With the exception of Transposition, none of the above items hold for material implication relative to $\models_4$. It is in particular the combined failures of Modus Ponens and the Deduction Theorem that have led to the development of alternative implication operators. $\mathcal{SROIQ}4$ and the other description logics described below use operators defined in the logics of Arieli and Avron [5,6]. The basic operator ($\supset$) is called *internal implication*, while *strong implication* ($\rightarrow$) is defined in terms of it. Equivalence ($\leftrightarrow$) is defined in turn using strong implication. For any truth-value assignment v, the semantics for the three are given below.

- $v(\mathcal{P} \supset \mathcal{Q}) = 1$ if $v(\mathcal{P}) \notin \{1, b\}$;
  $v(\mathcal{P} \supset \mathcal{Q}) = v(\mathcal{Q})$ otherwise.
- $v(\mathcal{P} \rightarrow \mathcal{Q}) = v((\mathcal{P} \supset \mathcal{Q}) \wedge (\neg\mathcal{Q} \supset \neg\mathcal{P}))$
- $v(\mathcal{P} \leftrightarrow \mathcal{Q}) = v((\mathcal{P} \rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \rightarrow \mathcal{P}))$

The semantics gives rise to the truth tables shown in Table 2. Internal implication satisfies Modus Ponens, the Deduction Theorem, Identity, and Supraclassicality, but *not* Modus Tollens, Transposition or Strong Equivalence. Strong implication satisfies all of these *except* (as shown below) the Deduction Theorem.

Example 1. *Let A and $\mathcal{P}$ be atomic formulas and $\mathcal{Q} = (A \wedge \neg A)$. Clearly, $\{\mathcal{Q}, \mathcal{P}\} \models_4 \mathcal{Q}$. However, an assignment v on which $v(A) = b$ and $v(\mathcal{P}) = 1$ shows $\{Q\} \not\models_4 \mathcal{P} \rightarrow \mathcal{Q}$.*

The failure of the Deduction Theorem for strong implication naturally leads one to ask whether there are any operators $\leadsto$ that satisfy Modus Ponens, Identity, Supraclassicality, Transposition, Modus Tollens, and Strong Equivalence, but *also* the Deduction Theorem. This is not the case.

Proposition 2. *If $\leadsto$ satisfies Modus Ponens and the Deduction Theorem, then $\leadsto$ satisfies none of Modus Tollens, Strong Equivalence, or Transposition.*

The above proposition holds whether or not Supraclassicality or Identity is satisfied.

Table 2

Truth tables for internal implication, strong implication, and equivalence.

$\supset$	n	0	1	b
n	1	1	1	1
0	1	1	1	1
1	n	0	1	b
b	n	0	1	b

$\to$	n	0	1	b
n	1	n	1	n
0	1	1	1	1
1	n	0	1	0
b	n	0	1	b

$\leftrightarrow$	n	0	1	b
n	1	n	n	n
0	n	1	0	0
1	n	0	1	0
b	n	0	0	b

Since strong implication satisfies Strong Equivalence, $\mathcal{P} \leftrightarrow \mathcal{Q}$ is a tautology *iff* the values of $\mathcal{P}$ and $\mathcal{Q}$ coincide in every interpretation. The analogous claim does *not* hold if equivalence were to be spelled out using $\supset$.

3. $\mathcal{SROIQ}$ and $\mathcal{SROIQ}4$

The syntax and semantics for classical $\mathcal{SROIQ}$ are given in Table 3. Below, N_C, N_R, and N_I are disjoint sets of *atomic concept names*, *atomic role names*, and *individual names*. Well-formed formulas are created from them, together with the connectives $\neg$, $\sqcap$, $\sqcup$, etc., and punctuation symbols. A $\mathcal{SROIQ}$ *knowledge base* is the union of a TBox, RBox, and ABox, defined below. The original presentation of $\mathcal{SROIQ}$ is found in [34].

Definition 3. *The set of* $\mathcal{SROIQ}$ role descriptions *is* $N_R \cup N_R^-$, *where* $N_R^- = \{R^- | R \in N_R\}$. *R and* R^- *are* inverses *of each other.* N_R *contains a* universal role U.

Definition 4. $R_1 \circ \ldots \circ R_n \sqsubseteq R$, *where* $n \geq 1$ *and* $R, R_i \in N_R \cup N_R^-$, *is a* role-inclusion axiom *(RIA). A* role-hierarchy *is a finite set of RIAs.*

Above, an expression such as $R_1 \circ \ldots \circ R_n$ is used to denote a composition of roles. A role R is *simple* if it: 1) does not appear on the right-hand side of an RIA; 2) is the inverse of a simple role; or 3) appears in the right-hand side of an RIA only if the left-hand side consists entirely of simple roles. Simplicity is needed to ensure decidability of $\mathcal{SROIQ}$.

Definition 5. $Ref(R)$, $Irr(R)$, *and* $Dis(R, S)$, *where R and S are roles other than U, are* role assertions. *A set of role assertions is simple* wrt *role-hierarchy H if each assertion* $Irr(R)$ *and* $Dis(R, S)$ *uses only roles that are simple* wrt H.

Ref, Irr, and Dis are used to specify that a role is reflexive, irreflexive, or disjoint with another role. Sym and $Trans$ may be used to express symmetry and transitivity, but since they can be encoded via other means, we will not discuss them.

Decidability also requires the role hierarchy to be *regular*.

Definition 6. *A strict partial order* $\prec$ *on* $N_R \cup N_R^-$ *is a* regular order iff *the following holds for all roles R and S:* $S \prec R$ iff $S^- \prec R$.

Definition 7. *Let* $\prec$ *be a regular order on roles. An RIA* $w \sqsubseteq R$ *is* $\prec$*-regular* iff $R \in R_N$ *and w has one of the below forms.*

1. $R \circ R$
2. R^-
3. $S_1 \circ \ldots \circ S_n$, *where each* $S_i \prec R$
4. $R \circ S_1 \circ \ldots \circ S_n$, *where each* $S_i \prec R$
5. $S_1 \circ \ldots \circ S_n \circ R$, *where each* $S_i \prec R$

A role hierarchy H is regular *if there exists a regular order* $\prec$ *such that each RIA in H is* $\prec$*-regular.*

Definition 8. *An* RBox *is a finite, regular role hierarchy H together with a finite set of role assertions simple* wrt H.

Definition 9. *If* $a_1, \ldots, a_n$ *are in* N_I, *then* $\{a_1, \ldots, a_n\}$ *is a* nominal. N_o *is the set of nominals.*

Definition 10. *The set of* $\mathcal{SROIQ}$-concept descriptions *is the smallest set such that:*

1. $\bot$, $\top$, *each* $C \in N_C$, *and each* $o \in N_o$ *is a concept description.*
2. *If C is a concept description, then* $\neg C$ *is a concept description.*
3. *If C and D are concept descriptions, R is a role description, S is a simple role description, and n is a nonnegative integer, then the following are all concept descriptions:*

$$(C \sqcap D),\quad (C \sqcup D),\quad \exists R.C,\quad \forall R.C,$$
$$\leq nS.C,\quad \geq nS.C,\quad \exists S.Self.$$

Table 3

Syntax and semantics of $\mathcal{SROIQ}$.

Concept	**Syntax**	**$\mathcal{SROIQ}$ Semantics**
atomic concept	$C \in N_C$	$C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
individual	$a \in N_I$	$a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$
nominal	$\{a_1, \ldots, a_n\}$, $a_i \in N_I$	$\{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\}$
role	$R \in N_R$	$R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
inverse role	R^-, $R \in N_R$	$R^{-\mathcal{I}} = \{(y,x) \vert (x,y) \in R^{\mathcal{I}}\}$
universal role	U	$U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
role composition	$R_1 \circ \ldots \circ R_n$	$\{(x,z) \vert (x,y_1) \in R_1^{\mathcal{I}} \wedge (y_1,y_2) \in R_2^{\mathcal{I}} \wedge \ldots$ $\ldots \wedge (y_n,z) \in R_n^{\mathcal{I}}\}$
top	$\top$	$\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$
bottom	$\bot$	$\bot^{\mathcal{I}} = \varnothing$
negation	$\neg C$	$(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}}/C^{\mathcal{I}}$
conjunction	$C_1 \sqcap C_2$	$(C_1 \sqcap C_2)^{\mathcal{I}} = C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
disjunction	$C_1 \sqcup C_2$	$(C_1 \sqcup C_2)^{\mathcal{I}} = C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}}$
exists restriction	$\exists R.C$	$\{x \vert (\exists y)[(x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}]\}$
value restriction	$\forall R.C$	$\{x \vert (\forall y)([(x,y) \in R^{\mathcal{I}} \mapsto y \in C^{\mathcal{I}}]\}$
self restriction	$\exists R.Self$	$\{x \vert (x,x) \in R^{\mathcal{I}}\}$
atmost restriction	$\leq nR.C$	$\{x \vert \sharp\{y \vert (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \leq n\}$
atleast restriction	$\geq nR.C$	$\{x \vert \sharp\{y \vert (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \geq n\}$

Axiom	**Syntax**	**$\mathcal{SROIQ}$ Semantics**
concept inclusion	$C_1 \sqsubseteq C_2$	$C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$
role inclusion	$R_1 \circ \ldots \circ R_n \sqsubseteq R_{n+1}$	$(R_1 \circ \ldots \circ R_n)^{\mathcal{I}} \subseteq R_{n+1}^{\mathcal{I}}$
reflexivity	$Ref(R)$	$\{(x,x) \vert x \in \Delta^{\mathcal{I}}\} \subseteq R^{\mathcal{I}}$
irreflexivity	$Irr(R)$	$\{(x,x) \vert x \in \Delta^{\mathcal{I}}\} \cap R^{\mathcal{I}} = \varnothing$
disjointness	$Dis(R,S)$	$S^{\mathcal{I}} \cap R^{\mathcal{I}} = \varnothing$
class assertion	$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
inequality assertion	$a \neq b$	$a^{\mathcal{I}} \neq b^{\mathcal{I}}$
role (instance) assertion	$R(a,b)$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$
negative role assertion	$\neg R(a,b)$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{I}}$

Definition 11. *If C and D are concept descriptions, then $C \sqsubseteq D$ is a* general concept inclusion *(GCI) axiom. A* TBox *is a finite set of GCIs.*

Definition 12. *If C is a concept description, $a, b \in N_I$, $R, S \in N_R \cup N_R^-$, and S is a simple role description, then $C(a)$, $R(a,b)$, $\neg S(a,b)$, and $a \neq b$, are* individual assertions. *A $\mathcal{SROIQ}$* ABox *is a set of individual assertions.*

We will call all GCIs, RIAs, role assertions, and individual assertions, *axioms*.

The semantics of $\mathcal{SROIQ}$ is classical. A *2-valued interpretation* (or *2-interpretation*) is a tuple $\langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, where $\Delta^{\mathcal{I}}$ is a nonempty set (the universe of discourse), and $\cdot^{\mathcal{I}}$ is a valuation function defined inductively as shown in Table 3. A 2-interpretation $\mathcal{I}$ 2-*satisfies* (is a 2-*model of*) an axiom A if the corresponding condition of $\mathcal{SROIQ}$ in Table 3 is met. $\mathcal{I}$ 2-satisfies (is a 2-model of) a knowledge base KB *iff* it 2-satisfies each axiom in KB. KB is 2-*satisfiable* (2-*unsatisfiable*) *iff* it has (does not have) a 2-model. KB *entails* an axiom A *wrt* $\mathcal{SROIQ}$ ($KB \models_{\mathcal{SROIQ}} A$) *iff* every 2-model of KB is a 2-model of A.

Since it is classical, $\mathcal{SROIQ}$ obeys the principle of explosion.

Example 13. *Let KB be a $\mathcal{SROIQ}$ knowledge base containing the following axioms:*

$Sedan(c435)$,
$Van(c435)$,
$Sedan \sqsubseteq \neg Van$,
$Incident(i90)$,
$involvedIn(c435\ i90)$,
$assignedTo(unit1\ i90)$,
$memberOf(tom\ unit1)$.

The knowledge base is intended to describe an accident involving a vehicle, the response team assigned to the accident, and the members of the team. The first three axioms are inconsistent, however, and because of this, unintuitive consequences such as $memberOf(c435\ unit1)$ and $Incident(tom)$ are entailed according to the classical $\mathcal{SROIQ}$ semantics.

To create the 4-valued logic $\mathcal{SROIQ}4$, we assign to each concept description both a *positive* extension P and a *negative* extension (*anti-extension*) N. $C(a)$ can thereby take on one of Belnap's four values.

1: $a^{\mathcal{I}} \in P$, $a^{\mathcal{I}} \notin N$.
0: $a^{\mathcal{I}} \notin P$, $a^{\mathcal{I}} \in N$.
n: $a^{\mathcal{I}} \notin P$, $a^{\mathcal{I}} \notin N$.
b: $a^{\mathcal{I}} \in P$, $a^{\mathcal{I}} \in N$.

Roles are given a 4-valued semantics as well (in earlier papers [46], we at times treated them classically). Specifically, $R^{\mathcal{I}}$ is $\langle P, N \rangle$, where P and N are subsets of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The reason for this treatment is to maintain paraconsistency while allowing Irr, Dis, Ref, $\exists R.Self$, and negative role assertions to be used in knowledge bases.

The syntax for $\mathcal{SROIQ}4$ differs from $\mathcal{SROIQ}$ in that it allows three types of inclusion axiom, based on the operators of Arieli and Avron's logic.

$C_1 \mapsto C_2$	*material inclusion*
$C_1 \sqsubset C_2$	*internal inclusion*
$C_1 \rightarrow C_2$	*strong inclusion*

Strictly speaking, axioms using the new inclusion operators are not defined in classical $\mathcal{SROIQ}$. However, we choose to extend the $\mathcal{SROIQ}$ syntax and semantics, stipulating that $(C \mapsto D)$, $(C \sqsubset D)$, and $(C \rightarrow D)$ have the same semantics in $\mathcal{SROIQ}$ as $(C \sqsubseteq D)$. We also stipulate that $(C \sqsubseteq D)$ is to be read as $(C \sqsubset D)$ in $\mathcal{SROIQ}4$. This allows a common language $\mathcal{L}$ for both $\mathcal{SROIQ}$ and $\mathcal{SROIQ}4$ knowledge bases and simplifies some of the proofs later in the paper.

If a knowledge base makes use of only $\sqsubseteq$, then we will call it a $\mathcal{SROIQ}$ knowledge base. Otherwise, it is a $\mathcal{SROIQ}4$ knowledge base. It should be noted that $\mathcal{SROIQ}$ encompasses several other description logics discussed in the literature. E.g, $\mathcal{ALC}$ is the logic defined using only N_C, N_R, N_I, and the connectives $\sqcap$, $\sqcup$, $\neg$, $\forall$, and $\exists$. $\mathcal{SHIQ}$ adds qualified cardinality restrictions.[1] As this is so, we can also speak of $\mathcal{ALC}4$ and $\mathcal{SHIQ}4$ knowledge bases. The paraconsistent framework defined here for $\mathcal{SROIQ}$ applies equally well to them.

The semantics of $\mathcal{SROIQ}4$ parallels that of its classical counterpart. A *4-valued interpretation* (*4-interpretation*) is a tuple $\langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, where $\Delta^{\mathcal{I}}$ is nonempty and $\cdot^{\mathcal{I}}$ is a valuation function defined inductively as shown in Table 4. We will also make use of the following notation: If C is a concept and $\mathcal{I}$ is a 4-interpretation such that $C^{\mathcal{I}} = \langle P, N \rangle$,

[1] $\mathcal{SHIQ}$ also allows an assertion of the form $Trans(R)$. However, this can be simulated in $\mathcal{SROIQ}$.

Table 4
Syntax and semantics of $\mathcal{SROIQ}4$.

Syntax	$\mathcal{SROIQ}4$ **Semantics**
$C \in N_C$	$C^{\mathcal{I}} = \langle P, N\rangle$, where $P, N \subseteq \Delta^{\mathcal{I}}$
$a \in N_I$	$a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$
$\{a_1, \ldots, a_n\}$, $a_i \in N_I$	$\langle\{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\}, N\rangle$, where $N \subseteq \Delta^{\mathcal{I}}$
$R \in N_R$	$R^{\mathcal{I}} = \langle P, N\rangle$, where $P, N \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
R^-, $R \in N_R$	$R^{-\mathcal{I}} = \langle P^-, N^-\rangle$, where $R^{\mathcal{I}} = \langle P, N\rangle$, $P^- = \{(b,a) \mid (a,b) \in P\}$, $N^- = \{(b,a) \mid (a,b) \in N\}$
U	$U^{\mathcal{I}} = \langle \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}, \varnothing\rangle$
$R_1 \circ \ldots \circ R_n$	$\langle\{(x,z) \mid (\exists y_{1\ldots n})[(x,y_1) \in p^+(R_1^{\mathcal{I}}) \wedge \ldots \wedge (y_n,z) \in p^+(R_n^{\mathcal{I}}]\})$ $\{(x,z) \mid (\forall y_{1\ldots n})[(x,y_1) \in p^-(R_1^{\mathcal{I}}) \vee \ldots \vee (y_n,z) \in p^-(R_n^{\mathcal{I}}]\})\rangle$
$\top$	$\top^{\mathcal{I}} = \langle \Delta^{\mathcal{I}}, \varnothing\rangle$
$\bot$	$\bot^{\mathcal{I}} = \langle \varnothing, \Delta^{\mathcal{I}}\rangle$
$\neg C$	$(\neg C)^{\mathcal{I}} = \langle N, P\rangle$, where $C^{\mathcal{I}} = \langle P, N\rangle$
$C_1 \sqcap C_2$	$(C_1 \sqcap C_2)^{\mathcal{I}} = \langle P_1 \cap P_2, N_1 \cup N_2\rangle$, where $C_i^{\mathcal{I}} = \langle P_i, N_i\rangle$
$C_1 \sqcup C_2$	$(C_1 \sqcup C_2)^{\mathcal{I}} = \langle P_1 \cup P_2, N_1 \cap N_2\rangle$, where $C_i^{\mathcal{I}} = \langle P_i, N_i\rangle$
$\exists R.C$	$\langle\{x \mid (\exists y)[(x,y) \in p^+(R^{\mathcal{I}}) \wedge y \in p^+(C^{\mathcal{I}})]\}$, $\{x \mid (\forall y)[(x,y) \in p^+(R^{\mathcal{I}}) \mapsto y \in p^-(C^{\mathcal{I}})]\}\rangle$
$\forall R.C$	$\langle\{x \mid (\forall y)([(x,y) \in p^+(R^{\mathcal{I}}) \mapsto y \in p^+(C^{\mathcal{I}})]\}$, $\{x \mid (\exists y)[(x,y) \in p^+(R^{\mathcal{I}}) \wedge y \in p^-(C^{\mathcal{I}})]\}\rangle$
$\exists R.Self$	$\langle\{x \mid (x,x) \in p^+(R^{\mathcal{I}})\}, N\rangle$, $N \subseteq \Delta^{\mathcal{I}}$
$\leq nR.C$	$\langle\{x \mid \sharp\{y \mid (x,y) \in p^+(R^{\mathcal{I}}) \wedge y \notin p^-(C^{\mathcal{I}})\} \leq n\}$, $\{x \mid \sharp\{y \mid (x,y) \in p^+(R^{\mathcal{I}}) \wedge y \in p^+(C^{\mathcal{I}})\} > n\}\rangle$
$\geq nR.C$	$\langle\{x \mid \sharp\{y \mid (x,y) \in p^+(R^{\mathcal{I}}) \wedge y \in p^+(C^{\mathcal{I}})\} \geq n\}$, $\{x \mid \sharp\{y \mid (x,y) \in p^+(R^{\mathcal{I}}) \wedge y \notin p^-(C^{\mathcal{I}})\} < n\}\rangle$

Syntax	$\mathcal{SROIQ}4$ **Semantics**
$C_1 \mapsto C_2$	$(\Delta^{\mathcal{I}} - p^-(C_1^{\mathcal{I}})) \subseteq p^+(C_2^{\mathcal{I}})$
$C_1 \sqsubset C_2$	$p^+(C_1^{\mathcal{I}}) \subseteq p^+(C_2^{\mathcal{I}})$
$C_1 \rightarrow C_2$	$p^+(C_1^{\mathcal{I}}) \subseteq p^+(C_2^{\mathcal{I}})$ and $p^-(C_2^{\mathcal{I}}) \subseteq p^-(C_1^{\mathcal{I}})$
$R_1 \circ \ldots \circ R_n \sqsubseteq R_{n+1}$	$p^+((R_1 \circ \ldots \circ R_n)^{\mathcal{I}}) \subseteq p^+(R_{n+1}^{\mathcal{I}})$
$Ref(R)$	$\{(x,x) \mid x \in \Delta^{\mathcal{I}}\} \subseteq p^+(R^{\mathcal{I}})$
$Irr(R)$	$\{(x,x) \mid x \in \Delta^{\mathcal{I}}\} \subseteq p^-(R^{\mathcal{I}})$
$Dis(R,S)$	$p^+(R^{\mathcal{I}}) \subseteq p^-(S^{\mathcal{I}})$ and $p^+(S^{\mathcal{I}}) \subseteq p^-(R^{\mathcal{I}})$
$C(a)$	$a^{\mathcal{I}} \in p^+(C^{\mathcal{I}})$
$a \neq b$	$a^{\mathcal{I}} \neq b^{\mathcal{I}}$
$R(a,b)$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in p^+(R^{\mathcal{I}})$
$\neg R(a,b)$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in p^-(R^{\mathcal{I}})$

then $p^+(C^{\mathcal{I}}) =_{def} P$ and $p^-(C^{\mathcal{I}}) =_{def} N$. A 4-interpretation 4-*satisfies* (is a 4-*model of*) an axiom A if the corresponding condition of $\mathcal{SROIQ}4$ in Table 4 is met. $\mathcal{I}$ 4-satisfies a knowledge base KB *iff* it 4-satisfies each axiom in KB. KB is 4-*satisfiable* (4-*unsatisfiable*) *iff* it has (does not have) a 4-model. KB *entails* an axiom A *wrt* $\mathcal{SROIQ}4$ ($KB \models_{\mathcal{SROIQ}4} A$) *iff* every 4-model of KB is a 4-model of A.

Example 14. *The following interpretation is a 4-model of the knowledge base in Example 13.*

$\Delta^{\mathcal{I}} = \{c435, i90, tom, unit1\}$
$c435^{\mathcal{I}} = c435$
$i90^{\mathcal{I}} = i90$
$tom^{\mathcal{I}} = tom$
$unit1^{\mathcal{I}} = unit1$
$Sedan^{\mathcal{I}} = Van^{\mathcal{I}} = \langle\{c435\}, \Delta^{\mathcal{I}}\rangle$
$Incident^{\mathcal{I}} = \langle\{i90\}, \Delta^{\mathcal{I}} - \{i90\}\rangle$
$involvedIn^{\mathcal{I}} = \langle\{(c435, i90)\}, (\Delta^{\mathcal{I}})^2 - \{(c435, i90)\}\rangle$
$assignedTo^{\mathcal{I}} = \langle\{(unit1, i90)\}, (\Delta^{\mathcal{I}})^2 - \{(unit1, i90)\}\rangle$
$memberOf^{\mathcal{I}} = \langle\{(tom, unit1)\}, (\Delta^{\mathcal{I}})^2 - \{(tom, unit1)\}\rangle$

The interpretation in Example 14 is a 4-model regardless of whether the single inclusion axiom of the knowledge base is read as material inclusion, internal inclusion, or strong inclusion. The interpretation is not a 4-model of the assertions $Incident(tom)$ and $memberOf(c435, unit1)$, however, and so the knowledge base does not entail them according to $\mathcal{SROIQ}4$.

It should be pointed out that internal inclusion $\sqsubset$ approximates $\rightarrow$, in the sense that the consequences obtainable using $\sqsubset$ are a subset of those obtainable using $\rightarrow$.

Proposition 15. *Let KB be a $\mathcal{SROIQ}4$ knowledge base, A a $\mathcal{SROIQ}4$ axiom, and KB' the knowledge base obtained by replacing one or more occurrences of $\sqsubset$ with $\rightarrow$. If $KB \models A$, then $KB' \models A$.*

In contrast, material inclusion $\mapsto$ does not approximate $\sqsubset$ in a similar way.

Example 16. *In general, $\{C \mapsto D\} \models C \mapsto D$ holds in $\mathcal{SROIQ}4$ but $\{C \sqsubset D\} \not\models C \mapsto D$ does not. This can be seen by modifying the interpretation in Example 14 so that $Sedan^{\mathcal{I}}$ and $Van^{\mathcal{I}}$ are defined to be $\langle\{c435\}, \{c435\}\rangle$. Under the modified interpretation, $Sedan \rightarrow \neg Van$ is satisfied but $Sedan \mapsto \neg Van$ is not.*

In $\mathcal{SROIQ}4$, inequality assertions $a \neq b$ have a classical semantics: $a \neq b$ is true if and only if $a^{\mathcal{I}} \neq b^{\mathcal{I}}$, and it is false otherwise. In other words, we have not given (in)equality a paraconsistent semantics. This is done to facilitate embedding $\mathcal{SROIQ}4$ into classical $\mathcal{SROIQ}$ so that classical tools can be used to reason paraconsistently. However, the use of classical semantics here entails the existence of $\mathcal{SROIQ}4$ knowledge bases that do not have 4-valued models. This is discussed in subsequent sections.

As in the classical setting, several equivalences involving negation hold in $\mathcal{SROIQ}4$.

Proposition 17. *For any $\mathcal{SROIQ}4$ concepts C, D and 4-interpretation $\mathcal{I}$ the following hold.*

1. $(\neg\top)^{\mathcal{I}} = \bot^{\mathcal{I}}$
2. $(\neg\bot)^{\mathcal{I}} = \top^{\mathcal{I}}$
3. $(\neg\neg C)^{\mathcal{I}} = C^{\mathcal{I}}$
4. $(\neg(C \sqcap D))^{\mathcal{I}} = (\neg C \sqcup \neg D)^{\mathcal{I}}$
5. $(\neg(C \sqcup D))^{\mathcal{I}} = (\neg C \sqcap \neg D)^{\mathcal{I}}$
6. $(\neg\exists R.C)^{\mathcal{I}} = (\forall R.\neg C)^{\mathcal{I}}$
7. $(\neg\forall R.C)^{\mathcal{I}} = (\exists R.\neg C)^{\mathcal{I}}$
8. $(\neg \leq nR.C)^{\mathcal{I}} = (\geq n+1R.C)^{\mathcal{I}}$
9. $(\neg \geq nR.C)^{\mathcal{I}} = (\leq n-1R.C)^{\mathcal{I}}$

In the classical DL setting, concept C is *subsumed* by concept D if and only if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for each classical interpretation $\mathcal{I}$, and C and D are equivalent if and only if $C^{\mathcal{I}} = D^{\mathcal{I}}$. These are easily related to satisfiability of GCIs. Since the semantics for $\mathcal{SROIQ}4$ is 4-valued, distinct notions of subsumption and equivalence naturally arise.

Definition 18. *Let C and D be $\mathcal{SROIQ}4$ concept descriptions and KB a $\mathcal{SROIQ}4$ knowledge base.*

1. *C is* 4-satisfiable wrt KB *if there is a 4-model $\mathcal{I}$ of KB such that $p^+(C^{\mathcal{I}})$ is not empty.*
2. *C is* weakly subsumed *by D (D* weakly subsumes *C) wrt KB if $p^+(C^{\mathcal{I}}) \subseteq p^+(D^{\mathcal{I}})$ in every 4-model $\mathcal{I}$ of KB.*
3. *C is* strongly subsumed *by D (D* strongly subsumes *C) wrt KB if $p^+(C^{\mathcal{I}}) \subseteq p^+(D^{\mathcal{I}})$ and $p^-(D^{\mathcal{I}}) \subseteq p^-(C^{\mathcal{I}})$ in every 4-model $\mathcal{I}$ of KB.*
4. *C and D are* weakly equivalent *($C \equiv D$) wrt KB iff $p^+(C^{\mathcal{I}}) = p^+(D^{\mathcal{I}})$ on each 4-model $\mathcal{I}$ of KB.*
5. *C and D are* strongly equivalent *($C \leftrightarrow D$) wrt KB iff $C^{\mathcal{I}} = D^{\mathcal{I}}$ on each 4-model $\mathcal{I}$ of KB.*

In Example 13, Van is 4-satisfiable relative to the knowledge base KB defined there, even though the only models for it assign common elements to both its positive and negative extensions. Similarly, since in every model of KB it holds that $p^+(Sedan^{\mathcal{I}}) \subseteq p^+((\neg Van)^{\mathcal{I}})$, regardless of the model $\mathcal{I}$ in question, $Sedan$ is weakly subsumed by $\neg Van$. It is not strongly subsumed, however. For instance, if the interpretation of Example 14 is modified so that $Sedan^{\mathcal{I}} = \langle\{c435\}, \{c435\}\rangle$ but $Van^{\mathcal{I}} = \langle\Delta^{\mathcal{I}}, \{c435\}\rangle$, then KB is still satisfied. However the criteria for strong subsumption are not met.

The below sets of claims follow immediately from Definition 18 and the semantics of $\mathcal{SROIQ}4$.

Proposition 19. *Let C and D be $\mathcal{SROIQ}4$ concept descriptions. The following all hold relative to any $\mathcal{SROIQ}4$ knowledge base KB.*

1. *C is 4-unsatisfiable iff C is weakly subsumed by $\bot$.*
2. *C and D are weakly equivalent iff C weakly subsumes D and D weakly subsumes C.*
3. *C and D are strongly equivalent iff C strongly subsumes D and D strongly subsumes C.*

Proposition 20. *Let C and D be $\mathcal{SROIQ}4$ concept descriptions. The following all hold relative to any $\mathcal{SROIQ}4$ knowledge base KB.*

1. *C is weakly subsumed by D* iff *$C \sqsubset D$ is satisfied on each $\mathcal{I}$.*
2. *C is strongly subsumed by D* iff *$C \to D$ is satisfied on each $\mathcal{I}$.*
3. *C and D are weakly equivalent iff $C \sqsubset D$ and $D \sqsubset C$ are both satisfied on each $\mathcal{I}$.*
4. *C and D are strongly equivalent iff $C \to D$ and $D \to C$ are both satisfied on each $\mathcal{I}$.*

3.1. $\mathcal{SROIQ}4$ is Sound

Every 2-interpretation $\mathcal{I}_2$ of a knowledge base KB corresponds to a 4-interpretation $\mathcal{I}_4$, which we will call the *4-counterpart* of $\mathcal{I}_2$:

- $\Delta^{\mathcal{I}_4} =_{def} \Delta^{\mathcal{I}_2}$.
- For each $a \in N_I$, $a^{\mathcal{I}_4} =_{def} a^{\mathcal{I}_2}$.
- For each $R \in N_R$, $R^{\mathcal{I}_4} =_{def} \langle R^{\mathcal{I}_2}, (\Delta^{\mathcal{I}_2} \times \Delta^{\mathcal{I}_2}) - R^{\mathcal{I}_2}\rangle$.
- For each $C \in N_C \cup N_o$, or $C = \exists R.Self$, $C^{\mathcal{I}_4} =_{def} \langle C^{\mathcal{I}_2}, \Delta^{\mathcal{I}_2} - C^{\mathcal{I}_2}\rangle$.

Since $\Delta^{\mathcal{I}_4} =_{def} \Delta^{\mathcal{I}_2}$, we will use Δ to refer to the common domain of discourse. In the counterpart $\mathcal{I}_4$, the positive and negative extensions of each atomic concept partition Δ. In fact, this holds for even non-atomic concept descriptions.

Proposition 21. *If $\mathcal{I}_2$ is a 2-interpretation and $\mathcal{I}_4$ its 4-counterpart, then for any concept description C, $C^{\mathcal{I}_4} = \langle C^{\mathcal{I}_2}, \Delta - C^{\mathcal{I}_2}\rangle$.*

Proposition 22. *If $\mathcal{I}_2$ is a 2-interpretation and $\mathcal{I}_4$ its 4-counterpart, then for any axiom A, $\mathcal{I}_4$ is a 4-model of A* iff *$\mathcal{I}_2$ is a 2-model of A.*

Given Proposition 22 and the fact that concept inclusion axioms have the same semantics in $\mathcal{SROIQ}$ regardless of their type, the following are all equivalent:

- $\mathcal{I}_2$ is a 2-model of $C \sqsubseteq D$.
- $\mathcal{I}_4$ is a 4-model of $C \sqsubset D$.
- $\mathcal{I}_4$ is a 4-model of $C \mapsto D$.
- $\mathcal{I}_4$ is a 4-model of $C \to D$.

Example 23. *Let KB_2 be the knowledge base from Example 13 with the assertion $Van(c435)$ removed. KB_2 is classically satisfiable, and we may define a 2-valued model $\mathcal{I}_2$ as follows:*

- $\Delta = \{c435, i90, tom, unit1\}$
- $c435^{\mathcal{I}_2} = c435$, $i90^{\mathcal{I}} = i90$, $tom^{\mathcal{I}_2} = tom$, $unit1^{\mathcal{I}} = unit1$
- $Sedan^{\mathcal{I}_2} = \{c435\}$
- $Van^{\mathcal{I}_2} = \varnothing$
- $Incident^{\mathcal{I}} = \{i90\}$
- $involvedIn^{\mathcal{I}} = \{(c435, i90)\}$
- $assignedTo^{\mathcal{I}} = \{(unit1, i90)\}$
- $memberOf^{\mathcal{I}} = \{(tom, unit1)\}$

The 4-valued counterpart $\mathcal{I}_4$ of $\mathcal{I}_2$ looks exactly like the 4-valued interpretation in Example 14, save that interpretations for Van and $Sedan$ are as shown below.

- $Sedan^{\mathcal{I}} = \langle\{c435\}, \Delta^{\mathcal{I}} - \{c435\}\rangle$
- $Van^{\mathcal{I}} = \langle\varnothing, \Delta^{\mathcal{I}}\rangle$

Given this, $(\neg Van)^{\mathcal{I}_4}$ is $\langle\Delta^{\mathcal{I}}, \varnothing\rangle$, which is in agreement with Proposition 21. Furthermore, it can be readily verified that $\mathcal{I}_4$ is a 4-model of KB_2, in agreement with Proposition 22.

Proposition 24. *Let KB be a $\mathcal{SROIQ}4$ knowledge base and A a $\mathcal{SROIQ}4$ axiom. If $KB \models_{\mathcal{SROIQ}4} A$, then $KB \models_{\mathcal{SROIQ}} A$.*

Proof. If $KB \models_{\mathcal{SROIQ}4} A$ and $\mathcal{I}_2$ 2-satisfies KB, then by Prop. 22 the 4-counterpart $\mathcal{I}_4$ of $\mathcal{I}_2$ 4-satisfies KB and hence A. Again by Prop. 22, $\mathcal{I}_2$ 2-satisfies A. $\square$

And so $\mathcal{SROIQ}4$ is sound *wrt* $\mathcal{SROIQ}$.

Example 25. *In Examples 13 and 23, one may conclude* $\neg Van(v435)$ *(which follows trivially by Modus Ponens). In Example 13, one may also conclude* $Van(435)$ *(by Reflexivity). In both cases, these conclusions are sound relative to classical* $\mathcal{SROIQ}$ *(in Example 13, since* KB *is classically inconsistent, everything follows from it).*

Significantly, regardless of what inclusion operator is used in $\mathcal{SROIQ}4$, if A is entailed in $\mathcal{SROIQ}4$, then it is also entailed according to $\mathcal{SROIQ}$. We can show this by replacing each inclusion symbol with $\sqsubseteq$.

Proposition 26. *Let* KB *be a* $\mathcal{SROIQ}4$ *knowledge base,* A *a* $\mathcal{SROIQ}4$ *axiom, and* KB' *and* A' *the* $\mathcal{SROIQ}$ *knowledge base and axiom obtained by replacing each occurrence of* $\sqsubset$, $\mapsto$, *and* $\rightarrow$ *with* $\sqsubseteq$.

$$\textit{If } KB \models_{\mathcal{SROIQ}4} A, \textit{ then } KB' \models_{\mathcal{SROIQ}} A'$$

Proof. If $KB \models_{\mathcal{SROIQ}4} A$ and $\mathcal{I}_2$ 2-satisfies KB', then by Prop. 22, the 4-counterpart $\mathcal{I}_4$ of $\mathcal{I}_2$ 4-satisfies KB'. Furthermore (by Prop. 22), $\mathcal{I}_4$ 4-satisfies KB. Since $KB \models_{\mathcal{SROIQ}4} A$, $\mathcal{I}_4$ 4-satisfies A, and so by Prop. 22, $\mathcal{I}_2$ 2-satisfies A. If $A \neq A'$, then A must be a $\mathcal{SROIQ}4$ inclusion axiom, which by convention has the same semantics in $\mathcal{SROIQ}$ as A'. And so $\mathcal{I}_2$ 2-satisfies A'. □

Example 27. *Consider the set* KB

$$\{Sedan(c435),\ Van(c435),\ Sedan \sqsubseteq \neg Van\}$$

of axioms from Example 13. One can conclude $\neg Van(c435)$ *but cannot conclude* $\neg Sedan(c435)$, *as internal inclusion does not permit Modus Tollens. However, the latter expression does follow if* $Sedan \sqsubset \neg Van$ *is replaced with* $Sedan \rightarrow \neg Van$. *In both cases, the conclusions are sound relative to classical* $\mathcal{SROIQ}$, *and they would remain so if, e.g.,* $Sedan(c435)$ *were removed in the second case to make the knowledge base consistent.*

Recall the two accounts of equivalence given earlier. Strong equivalence corresponds most closely to the classical account, and it is the account used in Arieli and Avron's logic. It is often more than one needs, however, since satisfaction and entailment in $\mathcal{SROIQ}4$ are defined using only the positive extensions of axioms. E.g.,

$$\{C(a)\} \models_{\mathcal{SROIQ}4} D(a) \text{ and}$$
$$\{D(a)\} \models_{\mathcal{SROIQ}4} C(a)$$

holds *iff* C and D are weakly equivalent. Furthermore, because $\mathcal{SROIQ}4$ is sound *wrt* $\mathcal{SROIQ}$, if C and D are even weakly equivalent in $\mathcal{SROIQ}4$, then the concepts are equivalent in classical $\mathcal{SROIQ}$. Something similar holds for disjointness. For instance, if

$$KB \models_{\mathcal{SROIQ}4} C \sqsubset \neg D \text{ or}$$
$$KB \models_{\mathcal{SROIQ}4} D \sqsubset \neg C,$$

then C and D are classically disjoint *wrt* KB.

4. Removing Gaps and Gluts

A paraconsistent logic typically rejects one or both of the below traditional laws.

- **Law of Noncontradiction (LNC)**: $\neg(\mathcal{P} \wedge \neg\mathcal{P})$
- **Law of the Excluded Middle (LEM)**: $\mathcal{P} \vee \neg\mathcal{P}$

$\mathcal{SROIQ}4$ rejects both.

Example 28. *Consider* $KB = \{\top(a)\}$. *In a classical setting, one could infer that* a *is a member of* $Person \sqcup \neg Person$. *However, in the 4-valued setting, this is not the case. E.g.,* $a^{\mathcal{I}} \notin p^{+}((Person \sqcup \neg Person)^{\mathcal{I}})$ *in any 4-valued interpretation* $\mathcal{I}$ *in which* $Person^{\mathcal{I}} = \langle \varnothing, \varnothing \rangle$. *Similarly, in the classical setting, one would be able to infer that* a *is not a member of* $Person \sqcap \neg Person$. *However,* $(Person \sqcap \neg Person)(a)$ *is satisfied in any 4-valued interpretation* $\mathcal{I}$ *in which* $Person^{\mathcal{I}} = \langle \Delta, \Delta \rangle$.

Nevertheless, inserting additional axioms into a $\mathcal{SROIQ}4$ knowledge base can selectively enforce the two laws, effectively making the knowledge base "more classical." In particular, if LEM is enforced but not LNC, then additional classical consequences can be drawn from the knowledge base while at the same time maintaining paraconsistency.

Following Arieli [3], we define the sets $EM(KB)$ ("excluded middle") and $EFQ(KB)$ ("ex falso quodlibet"). In any 4-model of $KB \cup EM(KB)$, LEM holds for concept assertions: For any individual a and atomic concept A, either $A(a)$ or $\neg A(a)$ is satisfied. Similarly, in any 4-model of $KB \cup EFQ(KB)$, LNC holds: At most one of $A(a)$ or $\neg A(a)$ is satisfied.

Definition 29. *Let* KB *be a* $\mathcal{SROIQ}4$ *knowledge base:*

– $EM(KB) =_{def} \{\top \sqsubset (A \sqcup \neg A) : A = \exists R.Self$ or $A \in N_C \cup N_o\}$.
– $EFQ(KB) =_{def} \{(A \sqcap \neg A) \sqsubset \bot : A = \exists R.Self$ or $A \in N_C \cup N_o\}$.

Proposition 30. *$\mathcal{I}$ is a 4-model of $EM(KB)$* iff *for each concept C of KB,*

$$p^+(C^{\mathcal{I}}) \cup p^-(C^{\mathcal{I}}) = \Delta^{\mathcal{I}}.$$

Proposition 31. *$\mathcal{I}$ is a 4-model of $EFQ(KB)$* iff *for each concept C of KB,*

$$p^+(C^{\mathcal{I}}) \cap p^-(C^{\mathcal{I}}) = \varnothing.$$

In [3], adding the analog of $EM(KB)$ to Arieli's logic yields Kleene's K_3, while adding the analog of $EFQ(KB)$ yields LP. The maneuver here can be seen as simulating similar logics in the DL setting. Both of these logics are deductively stronger than $\mathcal{SROIQ}4$. Since the purpose of $\mathcal{SROIQ}4$ is to reason paraconsistently, adding $EM(KB)$ to a knowledge base brings $\mathcal{SROIQ}4$ closer to $\mathcal{SROIQ}$ without abandoning that objective.

Adding both $EM(KB)$ and $EFQ(KB)$ allow one to simulate—to a point—$\mathcal{SROIQ}$ reasoning in $\mathcal{SROIQ}4$. Let $\mathcal{I}_4$ be a 4-interpretation of KB. We define a corresponding 2-interpretation $\mathcal{I}_2$ as shown below. Only the positive extensions are used in the construction.

– $\Delta^{\mathcal{I}_2} =_{def} \Delta^{\mathcal{I}_4}$.
– For each individual $a \in N_I$, $a^{\mathcal{I}_2} =_{def} a^{\mathcal{I}_4}$.
– For each role $R \in N_R$, $R^{\mathcal{I}_2} =_{def} p^+(R^{\mathcal{I}_4})$.
– For each $C \in N_C \cup N_o$, $C^{\mathcal{I}_2} =_{def} p^+(C^{\mathcal{I}_4})$.

Proposition 32. *$\mathcal{I}_4$ is a 4-model of $EM(KB) \cup EFQ(KB)$* iff *for each concept C,*

$$C^{\mathcal{I}_4} = \langle C^{\mathcal{I}_2}, \Delta - C^{\mathcal{I}_2} \rangle.$$

Proposition 33. *Let $\mathcal{I}_4$ be a 4-model of $EM(KB) \cup EFQ(KB)$. If A is a $\mathcal{SROIQ}4$ axiom of KB and not of the form $\neg R(a,b)$, $Irr(R)$, or $Dis(R,S)$, then $\mathcal{I}_2$ is a 2-model of A* iff *$\mathcal{I}_4$ is a 4-model of A.*

Proposition 34. *If KB is a $\mathcal{SROIQ}4$ knowledge base lacking axioms of the form $\neg R(a,b)$, $Irr(R)$, or $Dis(R,S)$, then*

$$KB \cup EM(KB) \cup EFQ(KB)$$

has a 4-model iff *KB has a 2-model.*

Example 35. *Consider the set KB*

$$\{Sedan(c435),\ Van(c435),\ Sedan \sqsubseteq \neg Van\}$$

of axioms from Example 13, and let $\mathcal{I}_4$ be defined as follows: $\Delta^{\mathcal{I}_4} = \{c435\}$, $c435^{\mathcal{I}_4} = c435$, and $Sedan^{\mathcal{I}_4} = Van^{\mathcal{I}_4} = \langle\{c435\},\{c435\}\rangle$. $\mathcal{I}4$ is a 4-valued model the axioms. However, it is not a 4-valued model of $(Van \sqcap \neg Van) \sqsubseteq \bot$, and so it is not a 4-valued model of $EFQ(KB)$.

The 2-valued counterpart $\mathcal{I}_2$ of $\mathcal{I}_4$ is: $\Delta^{\mathcal{I}_2} = \{c435\}$, $c435^{\mathcal{I}_2} = c435$, $Sedan^{\mathcal{I}_2} = Van^{\mathcal{I}_2} = \{c435\}$. This is not a 2-valued model of KB (the knowledge base has no such models). As implied by Proposition 34, $KB \cup EFQ(KB)$ has no 4-valued models. This can be seen by noting that $c435^{\mathcal{I}} \in p^+(Van^{\mathcal{I}}) \cap p^-(Van^{\mathcal{I}})$ in any 4-valued model $\mathcal{I}$ of KB.

Propositions 33 and 34 do not allow the role assertions above because we cannot add axioms for roles similar to $EM(KB)$ and $EFQ(KB)$. Such constructions involving roles are not allowed in $\mathcal{SROIQ}$ or $\mathcal{SROIQ}4$. If roles are taken as bivalent, then obviously the ban can be lifted.[2]

5. Embedding $\mathcal{SROIQ}4$ into $\mathcal{SROIQ}$

To allow the use of classical DL reasoners with $\mathcal{SROIQ}4$, we provide a translation function π (Table 5) from $\mathcal{SROIQ}4$ into $\mathcal{SROIQ}$. Each atomic concept C in a $\mathcal{SROIQ}4$ knowledge base is left untouched, but the negated concept $\neg C$ becomes the new atomic concept C'. In this way the 4-satisfiable concept $C \sqcap \neg C$ becomes the 2-satisfiable $C \sqcap C'$.[3] Below, $\mathcal{L}$ is used to refer to the language of $\mathcal{SROIQ}4$, and $\mathcal{L}'$ will be used to refer to the language created by adding primed counterparts to all atomic roles and concepts in $\mathcal{L}$. We also add new atomic concepts $C_{R.Self}$ for each R in $\mathcal{L}$, and C_o for each nominal $o \in N_o$. These will be used to stand for the negations of $R.Self$ and nominals, respectively.

It is clear that if KB is a $\mathcal{SROIQ}4$ knowledge base, then $\pi(KB)$ is a $\mathcal{SROIQ}$ knowledge

[2] Alternatively, one could attempt to address this issue by enhancing the language with further constructs for roles, as is done in [65]. However, the details of this remain to be spelled out.

[3] The basic approach is based on a method of translating paraconsistent propositional logics into classical logic [7,3].

base. Furthermore, since the transformation does not syntactically affect role inclusion axioms, if the role hierarchy is regular in KB, then it will be regular in $\pi(KB)$. Similarly, the counterpart of a simple role in KB is itself simple in $\pi(KB)$. As this is so, it is assured that π does not lead to undecidability.

As equality between individuals is interpreted classically in $\mathcal{SROIQ}4$, the most obvious translation of cardinality restrictions sometimes leads to unnecessary inconsistencies. Example 36 below illustrates how π avoids it. Section 6 shows that nominals cause a similar difficulty. In that case, however, the inconsistencies cannot be removed.

Example 36 ([46], Ex. 1)**.** *The assertion*

$$((\geq 2R.C) \wedge (\leq 1R.C))(a)$$

is 4-satisfiable but not 2-satisfiable. If the following scheme is used to translate cardinality restrictions

$$\begin{aligned}&\pi(\leq nR.C) = \leq nR.\pi(C)\\&\pi(\neg \leq nR.C) = \geq (n+1)R.\pi(C)\\&\pi(\geq nR.C) = \geq nR.\pi(C)\\&\pi(\neg \geq nR.C) = \leq (n-1)R.\pi(C)\end{aligned}$$

then the assertion remains unchanged and so classically unsatisfiable. The actual translation π, however, yields the 2-satisfiable $(\geq 2R.C) \wedge (\leq 1R.\neg C')$. While concepts $\neg C'$ and C intuitively refer to the same thing, they are formally distinct.

In general, the relationship between $K \models \mathcal{P}$ and the unsatisfiability of $K \cup \{\neg\mathcal{P}\}$ does not hold in the paraconsistent context, and so entailment cannot be reduced to satisfiability checking. Nevertheless, there is a strong and useful relationship between KB and $\pi(KB)$ that allows the use of satisfiability. If KB entails A in $\mathcal{SROIQ}4$, then $\pi(KB)$ entails $\pi(A)$ in $\mathcal{SROIQ}$ (Proposition 39). From this, it follows that $\pi(KB) \cup \{\neg\pi(A)\}$ is classically inconsistent.

Let $\mathcal{L}$ be the language of $\mathcal{SROIQ}4$. If $\mathcal{I}$ is a 4-interpretation of $\mathcal{L}$, then define the 2-interpretation $\mathcal{I}'$ (the primed counterpart of $\mathcal{I}$) over $\mathcal{L}'$ as follows:

- $\Delta^{\mathcal{I}'} =_{def} \Delta^{\mathcal{I}}$.
- For each individual $a \in N_I$, $a^{\mathcal{I}'} =_{def} a^{\mathcal{I}}$.
- For each role $R \in N_R \cup N_R^-$, $R^{\mathcal{I}'} =_{def} p^+(R^{\mathcal{I}})$ and $R'^{\mathcal{I}'} =_{def} p^-(R^{\mathcal{I}})$.
- For each atomic concept $C \in N_C$, $C^{\mathcal{I}'} =_{def} p^+(C^{\mathcal{I}})$ and $C'^{\mathcal{I}'} =_{def} p^-(C^{\mathcal{I}})$.
- For each nominal $o \in N_O$, $o^{\mathcal{I}'} =_{def} p^+(C^{\mathcal{I}})$, and $C_o^{\mathcal{I}'} =_{def} p^-(o^{\mathcal{I}})$.
- For each role $R \in N_R \cup N_R^-$, $(\exists R.Self)^{\mathcal{I}'} =_{def} p^+((\exists R.Self)^{\mathcal{I}})$, and
- For each role $R \in N_R \cup N_R^-$, $C_{R.Self}^{\mathcal{I}'} =_{def} p^-((\exists R.Self)^{\mathcal{I}})$.

It is clear that there is a 1–1 correspondence between the 4-interpretations of $\mathcal{L}$ and the 2-interpretations of $\mathcal{L}'$.

Proposition 37. *For any 4-interpretation $\mathcal{I}$ with primed counterpart $\mathcal{I}'$, and $\mathcal{SROIQ}4$ concept C, $p^+(C^{\mathcal{I}}) = \pi(C)^{\mathcal{I}'}$ and $p^-(C^{\mathcal{I}}) = \pi(\neg C)^{\mathcal{I}'}$ both hold.*

Proposition 38. *For any 4-interpretation $\mathcal{I}$, $\mathcal{I}$ is a 4-model of $\mathcal{SROIQ}4$ axiom A* iff *its primed counterpart $\mathcal{I}'$ is a 2-model of $\pi(A)$.*

Proposition 39. *For any $\mathcal{SROIQ}4$ knowledge base KB and axiom A, $KB \models_{\mathcal{SROIQ}4} A$* iff *$\pi(KB) \models_{\mathcal{SROIQ}} \pi(A)$.*

It is Proposition 39 that shows that reasoning with $\mathcal{SROIQ}4$ can be performed using the translation π and a classical reasoner.

Example 40. *Consider the set $\{Sedan(c435), Van(c435), Sedan \sqsubset \neg Van, Sedan \sqsubseteq Vehicle\}$. In any 4-valued model $\mathcal{I}$, it must be the case that $c435^{\mathcal{I}} \in p^+(Sedan^{\mathcal{I}})$ and $c435^{\mathcal{I}} \in p^+(Van^{\mathcal{I}})$, and also $c435^{\mathcal{I}} \in p^+((\neg Van)^{\mathcal{I}})$ and $c435^{\mathcal{I}} \in p^+(Vehicle^{\mathcal{I}})$. As such, $c435^{\mathcal{I}'} \in Sedan^{\mathcal{I}'}$, $c435^{\mathcal{I}'} \in Van^{\mathcal{I}'}$, $c435^{\mathcal{I}'} \in Sedan^{\mathcal{I}'}$, and $c435^{\mathcal{I}'} \in Van'^{\mathcal{I}'}$, where $\mathcal{I}'$ is the primed-counterpart of $\mathcal{I}$ and Van' is $\pi(\neg Van)$. Note that $\pi(Sedan \sqsubset \neg Van) = Sedan \sqsubseteq Van'$, and so it is clear that $\mathcal{I}'$ is a 2-valued model of $\pi(Sedan \sqsubset \neg Van)$. It is trivial to determine that $\mathcal{I}'$ must be a 2-valued model of $\pi(KB)$ (Proposition 38).*

In agreement with Proposition 37, $p^+(Van^{\mathcal{I}}) = Van^{\mathcal{I}'} = \pi(Van)^{\mathcal{I}'}$ and $p^-(Van^{\mathcal{I}}) = \pi(\neg Van)^{\mathcal{I}'}$. Also, $Vehicle(c435)$ is entailed both by the original knowledge base under the 4-valued semantics and by the translated knowledge base under the 2-valued semantics, in accordance with Proposition 39.

6. Partial Paraconsistency

$\mathcal{SROIQ}4$ is intended to avoid the explosions caused by inconsistencies. Unfortunately, some

Table 5
Translation of $\mathcal{SROIQ}4$ into $\mathcal{SROIQ}$.

$\pi(C) = C$, where $C \in N_C$ $\pi(o) = o$, where $o \in N_o$ $\pi(\top) = \top$ $\pi(\bot) = \bot$	$\pi(\neg C) = C'$, where C' new $\pi(\neg o) = C_o$, $o \in N_o$ and C_o new $\pi(\neg\neg C) = \pi(C)$ $\pi(\neg\top) = \bot$ $\pi(\neg\bot) = \top$
$\pi(E \sqcap D) = \pi(E) \sqcap \pi(D)$ $\pi(E \sqcup D) = \pi(E) \sqcup \pi(D)$	$\pi(\neg(E \sqcap D)) = \pi(\neg E) \sqcup \pi(\neg D)$ $\pi(\neg(E \sqcup D)) = \pi(\neg E) \sqcap \pi(\neg D)$
$\pi(\forall R.C) = \forall R.\pi(C)$ $\pi(\exists R.C) = \exists R.\pi(C)$ $\pi(\exists R.Self) = \exists R.Self$	$\pi(\neg(\forall R.C)) = \exists R.\pi(\neg C)$ $\pi(\neg(\exists R.C)) = \forall R.\pi(\neg C)$ $\pi(\neg\exists R.Self) = C_{R.Self}$
$\pi(\leq nR.C) = \leq nR.\neg\pi(\neg C)$ $\pi(\geq nR.C) = \geq nR.\pi(C)$	$\pi(\neg \leq nR.C) = \geq (n+1)R.\pi(C)$ $\pi(\neg \geq nR.C) = \leq (n-1)R.\neg\pi(\neg C)$
$\pi(C(a)) = \pi(C)(a)$ $\pi(R(a,b)) = R(a,b)$ $\pi(Ref(R)) = Ref(R)$ $\pi(Dis(R,S)) = \{R \sqsubseteq S', S \sqsubseteq R'\}$	$\pi(a \neq b) = (a \neq b)$ $\pi(\neg R(a,b)) = R'(a,b)$ $\pi(Irr(R)) = Ref(R')$
$\pi(C \mapsto D) = \neg\pi(\neg C) \sqsubseteq \pi(D)$ $\pi(C \sqsubset D) = \pi(C) \sqsubseteq \pi(D)$ $\pi(C \to D) = \{\pi(C \sqsubset D), \pi(\neg D \sqsubset \neg C)\}$ $\pi(w \sqsubseteq R_{n+1}) = w \sqsubseteq R_{n+1}$, with $w = R_1 \circ \ldots \circ R_n$	*material inclusion* *internal inclusion* *strong inclusion* *role inclusion*

$\mathcal{SROIQ}4$ knowledge bases—e.g., $\{(\top \sqcap \bot)(a)\}$—have no 4-models, and so $\mathcal{SROIQ}4$ is only "partially" paraconsistent. In some cases, inconsistent knowledge bases can be re-written into a classically equivalent form that avoids explosion. However, this is not the case in general.

In [50], it was reported that consistency can be maintained in $\mathcal{SHIQ}4$ by replacing $\top$ and $\bot$ with classically equivalent formulas. Specifically, let $SF(KB)$ be the knowledge base obtained by replacing each $\top$ with $A \sqcup \neg A$ and each $\bot$ with $A \sqcap \neg A$ (where A is a new atomic concept). $SF(KB)$ is guaranteed to possess a 4-model. However, nominals can conflict with cardinality restrictions, and so the analogous claim does not hold for $\mathcal{SROIQ}4$. For instance, any assertion of the form $\geq n+1R.\{a_1, \ldots, a_n\}(b)$ has no 4-valued models.[4] More simply, nominals can directly conflict with inequality assertions.

Example 41. *Suppose that the unique name assumption is enforced in a given knowledge base for days of the week (using axioms monday $\neq$ tuesday, tuesday $\neq$ wednesday, etc.). Then the following set of assertions has no 4-valued models.*

$$AvailableDay \subseteq \{monday, wednesday\},$$
$$AvailableDay(tuesday)$$

In order to maintain satisfiability, either cardinality restrictions (and inequality assertions with them) or nominals must go.

Proposition 42. *If C is a concept description in which nominals, $\top$, and $\bot$ do not appear, then C is 4-satisfiable.*

Proof. Let n be the largest integer used in an $\geq nR.C$ restriction. Let $C^{\mathcal{I}} = \langle \Delta, \Delta \rangle$ for each $C \in N_C$, where $|\Delta| = n+1$. For each $R \in N_R$, let $R^{\mathcal{I}} = \langle \Delta^2, \Delta^2 \rangle$ and $(\exists R.Self)^{\mathcal{I}} = \langle \Delta, \Delta \rangle$. We induct on a the degree of C. If $C \in N_C$, the claim clearly holds. Examining the $\mathcal{SROIQ}4$ semantics shows that where C is $\neg D$, $D \sqcup E$, $D \sqcap E$, $\forall R.D$, $\exists R.D$, $\geq nR.D$, or $\leq nR.D$, $C^{\mathcal{I}} = \langle \Delta, \Delta \rangle$. □

Proposition 43. *If KB is a $\mathcal{SROIQ}4$ knowledge base in which inequality assertions and nominals do not appear, then $SF(KB)$ has a 4-model.*

Proof. We use the interpretation $\mathcal{I}$ above. For simple assertions $C(a)$ and $R(a,b)$, the claim obvi-

[4] This is so regardless of whether nominals are interpreted classically or as done here.

ously holds. Since the anti-extension of each role R is Δ^2, $\neg R(a,b)$ also holds (we here reasonably assume that $R \neq U$), as do $Ref(R)$, $Irr(R)$, and $Dis(R,S)$. Since for all concepts C and D, $C^{\mathcal{I}} = D^{\mathcal{I}} = \langle \Delta, \Delta \rangle$, it follows that $C \sqsubset D$ and $C \to D$ hold. For $C \mapsto D$, observe that $\Delta^{\mathcal{I}} - p^{-}(C^{\mathcal{I}}) = \varnothing$, and so $C \mapsto D$ must be true in $\mathcal{I}$. The cases for RIAs are similar to $C \sqsubset D$. □

Proposition 44. *If C is a $\mathcal{SROIQ}4$ concept description in which $\leq$, $\geq$, $\top$, and $\bot$ do not appear, then C is 4-satisfiable.*

Proof. Let $\Delta^{\mathcal{I}} = \{d\}$, $C^{\mathcal{I}} = \langle \{d\}, \{d\} \rangle$ for each $C \in N_C \cup N_o$, and $a^{\mathcal{I}} = d$ for each $a \in N_I$. For each $R \in N_R$, let $R^{\mathcal{I}} = \langle \{(d,d)\}, \{(d,d)\} \rangle$ and $(\exists R.Self)^{\mathcal{I}} = \langle \{d\}, \{d\} \rangle$. Inducting on the degree of C, if $C \in N_C \cup N_o$ or $C = \exists R.Self$, then the claim holds. Examining the semantics of the concept description operators shows that the claim holds for $\neg C$, $C \sqcup D$, $C \sqcap D$, $\exists R.C$, and $\forall R.C$. □

Proposition 45. *If KB is a $\mathcal{SROIQ}4$ knowledge base in which inequality assertions, $\leq$, and $\geq$, do not appear, then $SF(KB)$ has a 4-model.*

Proof. Using $\mathcal{I}$ above, consider each type of axiom. For $C(a)$ and $R(a,b)$, the claim obviously holds. Since $p^{-}(R^{\mathcal{I}}) = \{(d,d)\}$, $\neg R(a,b)$ also holds (we assume that $R \neq U$). It is clear that $Ref(R)$, $Irr(R)$, and $Dis(R,S)$ also hold. Since for all concepts C and D, $C^{\mathcal{I}} = D^{\mathcal{I}} = \langle \{d\}, \{d\} \rangle$, $C \sqsubset D$ and $C \to D$ hold. For $C \mapsto D$, $\Delta^{\mathcal{I}} - p^{-}(C^{\mathcal{I}}) = \varnothing$, and so $\mathcal{I}$ 4-satisfies $C \mapsto D$. The cases for RIAs proceed similarly. □

One obvious solution to the conflict between cardinality restrictions and inequality assertions, on the one hand, and nominals on the other is to use *pseudo-nominals* in place of real nominals. E.g., instead of specifying the days of the week with the nominal $\{monday, tuesday, \ldots\}$, we represent the individuals with classes $Monday \sqcup Tuesday \sqcup \ldots$. In practice, this is sometimes done. However, it is well known that the ontology obtained via the substitution is not classically equivalent to the original, and furthermore that the use of pseudo-nominals leads to intuitively incorrect results. It's the same in the 4-valued context. However, to achieve paraconsistency, something of the classical logic must be abandoned, and for many applications, the loss of real nominals might be acceptable.

7. Tractable DLs

The latest revision of the Web Ontology Language (OWL 2) [30] features *profiles* having polynomial time complexities [57]. Below, we examine the tractable languages upon which these profiles are based. In particular, we examine $\mathcal{EL}^{++}$, which corresponds to OWL 2 EL; DL-Lite, which corresponds to OWL 2 QL; and Horn-$\mathcal{SHOIQ}$, which is an extension of OWL 2 RL. We focus on whether reasoning with the four-valued semantics presented earlier can preserve the tractability of these logics. Specifically, we focus on whether the reduction to classical logics yields formulas still within a tractable logic. It turns out that it does, provided internal inclusion is used. For material and strong inclusion, however, the reduction produces formulas outside of the tractable fragment.

7.1. $\mathcal{EL}^{++}$

The syntax of $\mathcal{EL}^{++}$, which restricts that of $\mathcal{SROIQ}$, is shown in Table 6. RIAs of the form

$$R \sqsubseteq S \text{ and } R_1 \circ R_2 \circ \ldots \circ R_n \sqsubseteq S,$$

where R_i, $S \in N_R$, are also allowed. Expressions involving concrete domains are permitted, too, but we do not consider them here. The semantics of $\mathcal{EL}^{++}$ is the same as that of $\mathcal{SROIQ}$.

The knowledge base $\{A \sqsubseteq \bot, A(a)\}$ shows that $\mathcal{EL}^{++}$ knowledge bases can be inconsistent. Inconsistency is caused specifically by the presence of $\bot$, and consistency can be restored by replacing each occurrence of $\bot$ with the classically equivalent $A_\bot \sqcap \neg A_\bot$ (where $A_\bot$ is a new atomic concept). Doing this is unproblematic: since the negation appears in front of an atomic concept, it will be eliminated when processed by π. In general, applying π to an $\mathcal{EL}^{++}$ concept (or a concept containing $\neg A$) produces an $\mathcal{EL}^{++}$ concept.

Unfortunately, the use of the paraconsistent inclusion operators causes problems. Applying π to $C \sqsubset D$ yields $\pi(C) \sqsubseteq \pi(D)$, where $\pi(C)$ and $\pi(D)$ are $\mathcal{EL}^{++}$ concepts. As such, $\pi(C) \sqsubseteq \pi(D)$ is still an $\mathcal{EL}^{++}$ axiom, and so internal inclusion does not destroy the tractability of $\mathcal{EL}^{++}$. However, applying π to GCIs using $\to$ and $\mapsto$ yield formulas not expressible in $\mathcal{EL}^{++}$, and so the guarantee of tractability is lost.

Example 46.

Table 6

$\mathcal{EL}^{++}$ and Horn-$\mathcal{SHOIQ}\circ$. The Horn-$\mathcal{SHOIQ}\circ$ normal form used is due to [43]. A, A_1, A_2, B are atomic concepts or nominals. $\mathcal{EL}^{++}$ ontologies may also contain RIAs.

Language	GCIs	Tractability Preservation
$\mathcal{EL}^{++}$	$C \sqsubseteq D$, where $C, D = \top \mid \bot \mid \{a\} \mid C_1 \sqcap C_2 \mid \exists r.C$	yes (only for $\sqsubset$)
Horn-$\mathcal{SHOIQ}\circ$	$\top \sqsubseteq A$, $A \sqsubseteq \bot$, $A_1 \sqcap A_2 \sqsubseteq B$, $\exists R.A \sqsubseteq B$, $A \sqsubseteq \exists R.B$, $A \sqsubseteq \forall S.B$, $A \sqsubseteq \geq nR.B$, $A \sqsubseteq \leq 1R.B$.	yes (only for $\sqsubset$)

$$\begin{aligned} \pi(A_1 \sqcap A_2 \mapsto B) &= \neg(A_1' \sqcup A_2') \sqsubseteq B \\ \pi(A_1 \sqcap A_2 \rightarrow B) &= \{A_1 \sqcap A_2 \sqsubseteq B, \\ &\quad B' \sqsubseteq (A_1' \sqcup A_2')\} \end{aligned}$$

Given the above discussion, the following theorem holds.

Proposition 47. *For any $\mathcal{EL}^{++}$ knowledge base KB and axiom α, $KB \models_4 \alpha$* iff *$\pi(KB) \models_2 \pi(\alpha)$. If all GCIs use $\sqsubset$, then $\pi(KB)$ is an $\mathcal{EL}^{++}$ knowledge base.*

Proof. The first claim holds in virtue of Proposition 39. Because applying π to any $\mathcal{EL}^{++}$ concept, RIA, or GCI $C \sqsubset D$ yields an $\mathcal{EL}^{++}$ expression, the second claim holds. □

Furthermore, since there is no negative role constructor to cause further inconsistencies in $\mathcal{EL}^{++}$, roles and RIAs can be interpreted classically (π would not alter them, anyway).

7.2. Horn-DLs

We focus on Horn-$\mathcal{SHOIQ}\circ$ [43]. The conclusions obtained for it hold for other Horn-DLs as well, including Horn-$\mathcal{SHOIQ}$ [56]. It's assumed that Horn-$\mathcal{SHOIQ}\circ$ ontologies are in the normal form [43] shown in Table 6.

All of the Horn-$\mathcal{SHOIQ}\circ$ concept constructors result in Horn-$\mathcal{SHOIQ}\circ$ expressions under π except $\leq 1R.B$, because $\pi(\leq 1R.B) =\leq 1R.\neg B'$. However, we can remain within Horn-$\mathcal{SHOIQ}\circ$ by post-processing the result of π as shown below.

- $\pi_{\mathrm{Horn}}(C) = C$, if C is

$$\top | \bot | A | A_1 \sqcap A_2 | \exists R.A | \forall S.B | \geq nR.B;$$

- $\pi_{\mathrm{Horn}}(\leq 1R.B) = \leq 1R.B^=$ (where $B^=$ is a new concept), with $B^= \sqcap B' \sqsubseteq \bot$ asserted;
- $\pi_{\mathrm{Horn}}(\neg(\leq 1R.B)) = \geq 2R.B$.

The last item is needed for processing inclusion axioms involving $\leq 1R.B$:

- $\pi_{\mathrm{Horn}}(A \mapsto \leq 1R.B) =$ $\{\neg A' \sqsubseteq \leq 1R.B^=, B^= \sqcap B' \sqsubseteq \bot\}$
- $\pi_{\mathrm{Horn}}(A \sqsubset \leq 1R.B) =$ $\{A \sqsubseteq \leq 1R.B^=, B^= \sqcap B' \sqsubseteq \bot\}$
- $\pi_{\mathrm{Horn}}(A \rightarrow \leq 1R.B) =$ $\{A \sqsubseteq \leq 1R.B^=, B^= \sqcap B' \sqsubseteq \bot, \geq 2R.B \sqsubseteq A'\}$

Given the above way of transforming axioms, $\pi_{\mathrm{Horn}}(A \sqsubset \leq 1R.B)$ is permissible, but the transformations for material and strong inclusion axioms are not. The example used for $\mathcal{EL}^{++}$ also works as a counterexample for the use of $\mapsto$ and $\rightarrow$ in Horn-$\mathcal{SHOIQ}\circ$. Since $A_1 \sqcap A_2 \sqsubseteq B$ is allowed in other Horn-DLs, the same conclusion holds for them. And so, if we wish to preserve the structure of tractable Horn-DLs, we must use only internal inclusion.

Proposition 48. *If KB is a Horn-$\mathcal{SHOIQ}\circ$ knowledge base using only $\sqsubset$-axioms, then it follows that $\pi_{Horn}(\pi(KB))$ is a Horn-$\mathcal{SHOIQ}\circ$ knowledge base.*

Given the revised π function, the conclusion obviously holds. The revised function also works for DLP [28], which corresponds to OWL 2 RL.

Observe that the additional transformations are sound, in the following sense.

Proposition 49. *Let $KB \cup \{A\}$ be a set of Horn-$\mathcal{SHOIQ}\circ$ assertions, with all GCIs being $\sqsubset$-axioms. If $\pi_{Horn}(\pi(KB)) \models_2 \pi_{Horn}(\pi(A))$, then $\pi(KB) \models_2 \pi(A)$.*

7.3. DL-Lite

The logics of the DL-Lite family are the maximal DLs supporting efficient query answering over large numbers of instances. The usual DL reasoning tasks performed in the DL-Lite family are known to be polynomial in the size of the KB (and LogSpace in the size of the ABox), and query answering is LogSpace in the size of the ABox [12]. Moreover, the DL-Lite family allows for separation between TBox and ABox reasoning during query evaluation. In particular, ABox reasoning can be carried out by an SQL engine [12].

We focus on two members of the DL-Lite family: DL-Lite$_{core}$, which forms the kernel of the family; and DL-Lite$_{\mathcal{R}}$, which corresponds to OWL 2 QL.

Concepts and roles of the DL-Lite family are formed using the grammar [12]

$$\begin{array}{ll} B ::= A \mid \exists R & R ::= P \mid P^- \\ C ::= B \mid \neg B & E ::= R \mid \neg R \end{array}$$

where $A \in N_C$, $P \in N_R$, and $P^- \in N_R^-$. Table 7 shows the syntax definitions of GCIs and RIAs.

A paraconsistent semantics similar to that described for $\mathcal{SROIQ}$ can be provided for DL-Lite$_{core}$ and DL-Lite$_{\mathcal{R}}$. Roles are 4-valued (Table 8), but $\exists R$ and $\neg\exists R$ are treated differently than in $\mathcal{SROIQ}$4. Intuitively, $p^+((\exists R)^\mathcal{I})$ is set of individuals x positively related to some y via R, while $p^-((\exists R)^\mathcal{I})$ is the set of individuals x negatively related to all y via R.

The knowledge base $\{B \sqsubseteq \neg A, B(a), A(a)\}$ is classically unsatisfiable in DL-Lite$_{core}$. $\{P_1 \sqsubseteq P_2, P_1 \sqsubseteq \neg P_2, P_1(a,b)\}$ is classically unsatisfiable in DL-Lite$_{\mathcal{R}}$. In the 4-valued context, however, every knowledge base is satisfiable.

Proposition 50. *Every DL-Lite$_{core}$ and DL-Lite$_{\mathcal{R}}$ knowledge base is 4-satisfiable.*

The DL-Lite π-transformations are given below. It is clear that the transformation of internal inclusion axioms stays within DL-Lite.

Definition 51. *Where R and E are roles, the DL-Lite π transformations are defined inductively.*

- $\pi_{Lite}(R \sqsubseteq E) = \pi_{Lite}(R) \sqsubseteq \pi_{Lite}(E)$;
- $\pi_{Lite}(\neg R) = R'$, *where R' is a new role name;*
- $\pi_{Lite}(\exists R) = \exists R$;
- $\pi_{Lite}(\neg\exists R) = \neg\exists R^=$, *where $R^=$ is a new role name and $R^= \sqsubseteq \neg R'$;*

For material inclusion and strong inclusion, because negated concepts are not allowed to occur on the left of a GCI, they do not preserve the DL-Lite structure. In contrast, the transformation of internal inclusion remains within DL-Lite. However, observe that the transformation

$$\pi_{\text{Lite}}(A \sqsubset \neg\exists R) = \{A \sqsubseteq \neg\exists R^=, R^= \sqsubseteq \neg R'\}$$

introduces a role inclusion axiom. In general, a DL-Lite$_{core}$ knowledge base might be transformed into a DL-Lite$_{\mathcal{R}}$ knowledge base. Regardless of this, the transformation preserves consequences.

Proposition 52. *If KB is a DL-Lite knowledge base and A an axiom (in which neither $\mapsto$ nor $\to$ are used),*

$$KB \models_4 A \text{ iff } \pi_{Lite}(KB) \models_{\textit{DL-Lite}_{\mathcal{R}}} \pi_{Lite}(A).$$

8. ELP

ELP [44] is a tractable rule language combining features of $\mathcal{EL}^{++}$ with Horn-like rules. Syntactically, ELP resembles Datalog or logic programming, in the sense that well-formed formulas are rules of the form $B \mapsto H$, where B and H are both conjunctions. In ELP, however, each conjunct is either a *concept atom* $C(t)$ or a role atom $R(t_1, t_2)$, where C is an $\mathcal{EL}^{++}$ concept description, R is an $\mathcal{EL}^{++}$ role, and each t is a *term* (an individual from N_I or else a variable from a set V of variables). Each rule is implicitly universally quantified (there are no free variables). To ensure decidability, it is assumed that there is a fixed set $V_s \subseteq V$ of *safe* variables; these range only over named elements of the domain of discourse.

Below, we present a paraconsistent semantics for ELP, defining what we call ELP4. The semantics is similar to one provided for a first-order logic defined by Lang [45], the primary differences being that we make use of internal implication $\supset$ where Lang uses his own implication operator, and we include safe variables (these could be omitted, however, yielding an undecidable logic). The rules described below confine themselves to what are essentially unary and binary predicates, but it is clear that the semantics can be easily modified to apply to rules using predicates of arbitrary arity. Furthermore, in order to ensure tractability, ELP rules conform to certain syntactic restrictions. We

Table 7
DL-Lite Family

Language	GCIs	Role Inclusions	Tractability preservation
DL-Lite$_{core}$	$B \sqsubseteq C$	$\varnothing$	yes (only for $\sqsubset$)
DL-Lite$_{\mathcal{R}}$	$B \sqsubseteq C$	$R \sqsubseteq E$	yes (only for $\sqsubset$)

Table 8
Four-valued semantics for expressions in DL-Lite

Syntax	**Semantics**
$\exists R$	$\langle \{x \mid \exists y, (x,y) \in p^+(R^{\mathcal{I}})\}, \{x \mid \forall y, (x,y) \in p^-(R^{\mathcal{I}})\} \rangle$
$\neg\exists R$	$\langle \{x \mid \forall y, (x,y) \in p^-(R^{\mathcal{I}})\}, \{x \mid \exists y, (x,y) \in p^+(R^{\mathcal{I}})\} \rangle$

do not discuss the restrictions below. However, the embedding into classical ELP maintains the syntactic structure of the rules, and so tractability is preserved. Also, since the restrictions are not part of the discussion, it is clear that most of the definitions and results below apply equally well to rule-bases that do not satisfy them.

As originally defined, a *rule base* for ELP consists of rules of the form $B \mapsto H$, where $\mapsto$ is just material implication. In moving to the 4-valued context, we also allow rules using internal ($\supset$) and strong ($\rightarrow$) implication. 4-valued interpretations for ELP rule bases are defined essentially as they were earlier. However, semantics for rules must be given. This is done by defining two forms of satisfaction: t-satisfaction ($\models_t$) and f-satisfaction ($\models_f$). The notation $\models_t$ and $\models_f$ is adopted from Lang [45].

Definition 53. *An element $d \in \Delta^{\mathcal{I}}$ is* named *in interpretation $\mathcal{I}$* iff *there is an $a \in N_I$ such that $a^{\mathcal{I}} = d$. Otherwise, d is* unnamed *in $\mathcal{I}$. A variable assignment $\sigma_{\mathcal{I}}$ for $\mathcal{I}$ is a function from $V \cup N_I$ to $\Delta^{\mathcal{I}}$ such that*

1. *if $t \in N_I$, then $\sigma_{\mathcal{I}}(t) = t^{\mathcal{I}}$;*
2. *if $t \in V_S$, then $\sigma_{\mathcal{I}}(t)$ is named in $\mathcal{I}$.*

If $\mathcal{I}$ is an interpretation and $\sigma_{\mathcal{I}}$ an assignment for $\mathcal{I}$, then $\sigma_{\mathcal{I}}[x/d]$ is a function from $V \cup N_I$ to $\Delta^{\mathcal{I}}$ such that for all $t \in N_I \cup V$,

1. *if $t \neq x$, then $\sigma_{\mathcal{I}}[x/d](t) = \sigma_{\mathcal{I}}(t)$;*
2. *otherwise, $\sigma_{\mathcal{I}}[x/d](t) = d$.*

Observe that if $x \in V_s$ but d is not named in $\mathcal{I}$, $\sigma_{\mathcal{I}}[x/d]$ is not a valid assignment.

Definition 54. *Let $\mathcal{I}$ be an interpretation and $\sigma_{\mathcal{I}}$ a variable assignment for $\mathcal{I}$.*

1. *If C is a concept description and t a term,*
   - $\mathcal{I}, \sigma_{\mathcal{I}} \models_t C(t)$ iff $\sigma_{\mathcal{I}}(t) \in p^+(C^{\mathcal{I}})$.
   - $\mathcal{I}, \sigma_{\mathcal{I}} \models_f C(t)$ iff $\sigma_{\mathcal{I}}(t) \in p^-(C^{\mathcal{I}})$.
2. *If R is a role and t_1 and t_2 terms,*
   - $\mathcal{I}, \sigma_{\mathcal{I}} \models_t R(t_1, t_2)$ iff $(\sigma_{\mathcal{I}}(t_1), \sigma_{\mathcal{I}}(t_2)) \in p^+(R^{\mathcal{I}})$.
   - $\mathcal{I}, \sigma_{\mathcal{I}} \models_f R(t_1, t_2)$ iff $(\sigma_{\mathcal{I}}(t_1), \sigma_{\mathcal{I}}(t_2)) \in p^-(R^{\mathcal{I}})$.
3. *For conjunction $(\mathcal{P} \wedge \mathcal{Q})$,*
   - *$\mathcal{I}, \sigma_{\mathcal{I}} \models_t (\mathcal{P} \wedge \mathcal{Q})$ iff $\mathcal{I}, \sigma_{\mathcal{I}} \models_t \mathcal{P}$ and $\mathcal{I}, \sigma \models_t \mathcal{Q}$,*
   - *$\mathcal{I}, \sigma_{\mathcal{I}} \models_f (\mathcal{P} \wedge \mathcal{Q})$ iff $\mathcal{I}, \sigma_{\mathcal{I}} \models_f \mathcal{P}$ or $\mathcal{I}, \sigma \models_f \mathcal{Q}$,*
4. *For material implication $(\mathcal{P} \mapsto \mathcal{Q})$,*
   - *$\mathcal{I}, \sigma_{\mathcal{I}} \models_t (\mathcal{P} \mapsto \mathcal{Q})$ iff $\mathcal{I}, \sigma_{\mathcal{I}} \models_f \mathcal{P}$ or $\mathcal{I}, \sigma_{\mathcal{I}} \models_t \mathcal{Q}$,*
   - *$\mathcal{I}, \sigma_{\mathcal{I}} \models_f (\mathcal{P} \mapsto \mathcal{Q})$ iff $\mathcal{I}, \sigma_{\mathcal{I}} \models_t \mathcal{P}$ and $\mathcal{I}, \sigma_{\mathcal{I}} \models_f \mathcal{Q}$,*
5. *For internal implication $(\mathcal{P} \supset \mathcal{Q})$,*
   - *$\mathcal{I}, \sigma_{\mathcal{I}} \models_t (\mathcal{P} \supset \mathcal{Q})$ iff $\mathcal{I}, \sigma_{\mathcal{I}} \not\models_t \mathcal{P}$ or $\mathcal{I}, \sigma_{\mathcal{I}} \models_t \mathcal{Q}$,*
   - *$\mathcal{I}, \sigma_{\mathcal{I}} \models_f (\mathcal{P} \supset \mathcal{Q})$ iff $\mathcal{I}, \sigma_{\mathcal{I}} \models_t \mathcal{P}$ and $\mathcal{I}, \sigma_{\mathcal{I}} \models_f \mathcal{Q}$,*
6. *For strong implication $(\mathcal{P} \rightarrow \mathcal{Q})$,*
   - *$\mathcal{I}, \sigma_{\mathcal{I}} \models_t (\mathcal{P} \rightarrow \mathcal{Q})$ iff $(\mathcal{I}, \sigma_{\mathcal{I}} \not\models_t \mathcal{P}$ or $\mathcal{I}, \sigma_{\mathcal{I}} \models_t \mathcal{Q})$ and $(\mathcal{I}, \sigma_{\mathcal{I}} \models_f \mathcal{P}$ or $\mathcal{I}, \sigma_{\mathcal{I}} \not\models_f \mathcal{Q})$.*
   - *$\mathcal{I}, \sigma_{\mathcal{I}} \models_f (\mathcal{P} \rightarrow \mathcal{Q})$ iff $\mathcal{I}, \sigma_{\mathcal{I}} \models_t \mathcal{P}$ and $\mathcal{I}, \sigma_{\mathcal{I}} \models_f \mathcal{Q}$.*
7. *For $(\forall x)\mathcal{P}$,*
   (a) *if $x \in V_s$,*
      - *$\mathcal{I}, \sigma_{\mathcal{I}} \models_t (\forall x)\mathcal{P}$ iff $\mathcal{I}, \sigma_{\mathcal{I}}[x/d] \models_t \mathcal{P}$ for all* named *$d \in \Delta^{\mathcal{I}}$,*
      - *$\mathcal{I}, \sigma_{\mathcal{I}} \models_f (\forall x)\mathcal{P}$ iff $\mathcal{I}, \sigma_{\mathcal{I}}[x/d] \models_f \mathcal{P}$ for some* named *$d \in \Delta^{\mathcal{I}}$.*

   (b) *if $x \notin V_s$,*

– $\mathcal{I}, \sigma_{\mathcal{I}} \models_t (\forall x)\mathcal{P}$ *iff* $\mathcal{I}, \sigma_{\mathcal{I}}[x/d] \models_t \mathcal{P}$ *for all* $d \in \Delta^{\mathcal{I}}$,
– $\mathcal{I}, \sigma_{\mathcal{I}} \models_f (\forall x)\mathcal{P}$ *iff* $\mathcal{I}, \sigma_{\mathcal{I}}[x/d] \models_f \mathcal{P}$ *for some* $d \in \Delta^{\mathcal{I}}$.

Definition 55. *Let $\mathcal{I}$ be an interpretation, $\mathcal{P}$ a well-formed formula of ELP, and K a set of such formulas.*

1. *$\mathcal{I}$ 4-satisfies $\mathcal{P}$ ($\mathcal{I}$ is a 4-model of $\mathcal{P}$)* iff *$\mathcal{I}, \sigma_{\mathcal{I}} \models_t \mathcal{P}$ for each assignment $\sigma_{\mathcal{I}}$.*
2. *K entails $\mathcal{P}$ ($K \models_{ELP4} \mathcal{P}$)* iff *every 4-model of K is a 4-model of $\mathcal{P}$.*

According to [44], every classical ELP rule base can be converted into an equisatisfiable one via the transformation rules shown in Table 9. Since there are three different types of rules, the analogous claim is problematic in the 4-valued context. Nevertheless, if rules are interpreted using $\supset$, conversion to normal form preserves 4-satsifiability.

Definition 56. *Let RB be an ELP rule base utilizing only $\supset$-rules. RB is in* normal form iff *for each rule $B \supset H$,*

1. *Each concept atom in B is of the form $C(t)$, where $C \in N_C \cup N_o \cup \{\top\}$,*
2. *Each atom in H is of the form $A(t)$, $\exists R.B(t)$, or $R(t, u)$, where*
   (a) *$A \in N_C \cup N_o \cup \{\bot\}$,*
   (b) *$B \in N_C$,*
   (c) *$R \in N_R$, and*
   (d) *$t, u \in V \cup N_I$.*
3. *Every variable in H appears in B.*

Proposition 57. *Let KB be a $\supset$-rule base and KB' the result of applying one of H1,…,H4, B1,…,B3 from Table 9 to KB. KB is 4-satisfiable* iff *KB' is.*

We define the sequence KB_0, KB_1, …, where $KB_0 = KB$ and KB_{n+1} is the result of applying one transformation rule to KB_n. Let KB_N be the first n such that $KB_n = KB_{n+1}$. KB_N is in normal form [44]. Given the above proposition, it is clear that KB_N is satisfiable if and only if KB is.

The transformation π can be extended in a rather simple fashion to embed ELP4 into classical ELP, provided $\supset$ is used. As with the tractable description logics, the other inclusion operators yield structures outside of the language (one could nevertheless use $\rightarrow$ and $\mapsto$ as well, translating them as indicated in Table 5. Doing so would not yield meaningless expressions, but the results would not constitute expressions of ELP). Since the structure of the rule base is left intact by the transformation, the syntactic constraints on ELP rules are met.

$$\begin{aligned}
&\pi(C(t)) = \pi(C)(t)\\
&\pi(R(t_1, t_2)) = \pi(R)(t_1, t_2)\\
&\pi(\mathcal{Q} \wedge \mathcal{R}) = \pi(\mathcal{Q}) \wedge \pi(\mathcal{R})\\
&\pi(\mathcal{Q} \vee \mathcal{R}) = \pi(\mathcal{Q}) \vee \pi(\mathcal{R})\\
&\pi(\mathcal{Q} \supset \mathcal{R}) = \pi(\mathcal{Q}) \mapsto \pi(\mathcal{R})\\
&\pi((\forall x)\mathcal{Q}) = (\forall x)\pi(\mathcal{Q})
\end{aligned}$$

Define $\pi(KB)$ as $\{\pi(R) | R \in KB\}$. Earlier, we defined the primed counterpart $\mathcal{I}'$ for each 4-interpretation $\mathcal{I}$, and we showed that for any $\mathcal{SROIQ}$ concept C, $p^+(C^{\mathcal{I}}) = \pi(C)^{\mathcal{I}'}$ and $p^-(C^{\mathcal{I}}) = \pi(\neg C)^{\mathcal{I}'}$. We show here that for any rule $r : B \supset H$, $\mathcal{I}$ 4-satisfies r *iff* $\mathcal{I}'$ 2-satisfies $\pi(r)$.[5]

Proposition 58. *If $\mathcal{P}$ is a well-formed formula of ELP, $\mathcal{I}$ is a 4-interpretation of $\mathcal{P}$, and $\mathcal{I}'$ is the primed counterpart of $\mathcal{I}$, then for any assignment $\sigma_{\mathcal{I}}$, $\mathcal{I}, \sigma_{\mathcal{I}} \models_t \mathcal{P}$* iff *$\mathcal{I}', \sigma_{\mathcal{I}'} \models \pi(\mathcal{P})$.*

From this, the preservation of consequences readily follows.

Proposition 59. *Let KB be an ELP rule base and $\mathcal{P}$ a well-formed formula of ELP. $KB \models_{ELP4} \mathcal{P}$* iff *$\pi(KB) \models_{ELP} \pi(\mathcal{P})$.*

Example 60. *The following set RB of (normal form) ELP rules is classically inconsistent.*

1. $motherOf(mary\ eve)$
2. $fatherOf(tom\ eve)$
3. $divorced(tom)$
4. $loves(x\ y) \supset happy(x)$
5. $motherOf(x\ y) \wedge fatherOf(z\ y) \supset loves(x\ z)$
6. $motherOf(x\ y) \wedge fatherOf(z\ y) \supset loves(z\ x)$
7. $divorced(x) \supset unhappy(x)$
8. $happy(x) \wedge unhappy(x) \supset \bot(x)$

Because of the occurrence of $\bot$, it is also inconsistent under the 4-valued semantics. This can be remedied by replacing rule 8 *with* $8'$ *(to form RB').*

8.′ $happy(x) \wedge unhappy(x) \supset A_\bot(x) \wedge \neg A_\bot(x)$.

Though the negation is not technically allowed in $\mathcal{EL}^{++}$, Upon transformation by π, $A_\bot(x) \wedge \neg A_\bot(x)$ becomes $A_\bot(x) \wedge A'_\bot(x)$, which is allowed. Apart

[5]Observe that the set of variables is fixed, regardless of whether one is talking about $\mathcal{I}$ or $\mathcal{I}'$; furthermore, the interpretation of individuals and variables is the same in both interpretations. And so "safety" is not affected.

Table 9

Converting a classical ELP rule $B \mapsto H$ to normal form (taken from [44]).

H1	If $(C \sqcap D)(t) \in H$, then replace it with $C(t) \wedge D(t)$.
H2	If $\top(x) \in H$, then delete it. If H would then be empty, delete r from KB.
H3	If H contains a variable x not occurring in B, add $\top(x)$ to B.
H4	If $\exists R.C(t) \in H$ with $C \notin N_C$, replace it in r with $\exists R.A(t)$ where A is new to KB, and add the rule $A(x) \mapsto C(x)$.
B1	If $(C \sqcap D)(t) \in B$, then replace it with $C(t) \wedge D(t)$.
B2	If $\bot(x) \in B$, delete r from KB.
B3	If $\exists R.C(t) \in B$, replace it with $R(t, x)$, $C(x)$, where x is new.

from this and the replacement of $\supset$ with $\mapsto$, the rule base is otherwise unaltered.[6]

We note that RB' is 4-satisfiable. Furthermore, $\pi(RB')$ is 2-satisfiable. In both, one may infer happy(tom) and unhappy(tom) (under the appropriate semantics). However, in neither case can one infer an arbitrary expression such as unhappy(mary).

If one were to attempt to transform the rule base using strong implication (according to Table 5), then rules 5, 6, and 8 would introduce disjunctions. For instance,

$5'_1$ $motherOf(x\ y) \wedge fatherOf(z\ y) \mapsto loves(x\ z)$
$5'_2$ $loves'(x\ z) \mapsto motherOf'(x\ y) \vee fatherOf'(z\ y)$.

This is outside of the rule language.

9. Empirical Evaluations

The last two sections indicate that, for some description logics and their rule-based extensions, internal inclusion must be used in order to guarantee tractability. However, even in cases where the translation does not lead to expressions of a strictly more expressive language—e.g., as in the case of $\mathcal{SROIQ}$—the choice of operator can have a significant effect on performance. While embedding internal inclusion axioms into a classical DL potentially leads to the creation of new concept names, it typically does not increase the number of inclusion axioms,[7] and the translation eliminates all occurrences of negation. In contrast, embedding strong inclusion axioms essentially doubles both the number of concept names and the number of inclusion axioms. Embedding material inclusion introduces negation into the left-hand-sides of axioms, and it potentially leads to the introduction of a greater number of new concepts than does internal inclusion. Though the size of the translation is linear with regard to the size of the original regardless of the inclusion operator used, these differences can have a significant impact and make reasoning with strong and material inclusion more difficult than reasoning with internal inclusion alone.

We have performed a set of empirical tests which, though tentative and limited in nature, do appear to support this claim. Several knowledge bases (either downloaded or else programmatically created) were translated according to the π function described in Section 5, and a DL reasoner (Pellet [67]) was used to process both the original knowledge bases and their translations. Specifically, the reasoner computed class and property hierarchies, determined class membership for individuals, and checked for consistency. Some but not all of the original ontologies were satisfiable.[8]

The translations were performed using a custom Java library implementing the π function and based on methods provided by the OWL API [33]. Three distinct translations of each knowledge base were created, each interpreting the inclusion axioms of the original according to one of the three new operators.

Two knowledge bases were created specifically for the tests using the Lehigh Benchmark (LUBM)

[6] Though not explicitly stated in [44], A rule of the form $B \mapsto C(t) \wedge D(t) \wedge ...$ can be split into two rules, $B \mapsto C(t) \wedge ...$ and $B \mapsto D(t) \wedge ....$ We will assume that such transformations are also performed when converting to normal form.

[7] This is not true for the paraconsistent versions of Horn DLs and DL-Lite, as discussed earlier; for those logics, new axioms are added.

[8] The number of unsatisfiable concepts in each ontology is: `amino-acid` (0), `pizza` (2), `proton` (15), `tambis-patched` (144), `university-1` (0), `university-2` (0), `wine` (0).

suite [27]. The suite provides a tool for generating ontologies populated with a variable number of universities, departments, professors, students, etc. For the evaluations, the default settings of the tool were used, but the resulting knowledge bases were manually edited.[9] Disjoint class axioms were also manually added (asserting, e.g., that assistant, associate, and full professors are disjoint classes). Inconsistent versions of the knowledge bases were also created by inserting same-as assertions (stating, for instance, that a particular student was identical to a publication).

For the experiments, each ontology was processed 21 times (without restarting between runs), and the time required to complete all computations was recorded. The first run—which involved memory allocation not occurring in subsequent runs—was ignored in each case. A 15 minute time-limit was used; runs requiring more time than that were terminated. All experiments were performed on a consumer desktop computer.

Tables 10 and 11 describe the knowledge bases evaluated. The relative sizes of the translations are given in Table 12. As shown there, translating according to strong inclusion typically increases the number of concept names by a factor of 2. The number of inclusion axioms is not doubled—it is assumed that this is due to the presence of duplicate axioms that are eliminated by the OWL API. As far as the concepts are concerned, factors greater than 2 can be attributed to the presence of nominals in the input. Factors less than 2 are also possible, as when concepts only appear in positive concept assertions. Using material inclusion or internal inclusion also increases the number of concepts. However, this number is smaller than when strong inclusion is used.

Table 13 displays the average time required to process the knowledge bases and their translations. When internal inclusion was used, the test system was in each case able to process the translation (the time required was significantly greater than that needed for the original, however). In contrast, if either strong or material inclusion was used, processing the translation was typically not possible. The reasoner either ran out of memory, or else execution was manually terminated due to the time-limit. The inability to process knowledge bases based on material inclusion can be explained by the introduction of negation into axioms. This does not occur when internal inclusion is used.

[9] The smaller knowledge base consisted of 5 departments, 25 research groups, 1000 (graduate and teaching) courses, 100 professors (lecturer, assistant, associate, full), 1000 students (graduate and undergraduate), 500 publications, and 200 teaching and research assistants. The larger instance contained roughly 2-8 times these numbers. In both cases, 10 universities were used.

10. Conclusions and Related Work

In this paper, we have described several paraconsistent description logics, including $\mathcal{SROIQ}4$, and we have shown that they are classically sound and that their embeddings into classical logics are consequence preserving. It should be pointed out that since the most recent version of OWL [30] is based in large part on $\mathcal{SROIQ}$, it is not unreasonable to suppose that $\mathcal{SROIQ}4$ has real practical value. Inconsistencies often arise in real world situations, and the theory surrounding $\mathcal{SROIQ}4$ allows the use of existing classical OWL inference engines to reason correctly even over inconsistent OWL knowledge bases.

The issue of handling inconsistency is a common one in the field of knowledge representation and reasoning, and also more generally in the context of data management. Beyond multivalued paraconsistent logics, several other approaches have been studied. For instance, regardless of whether they are paraconsistent or not, the various nonmonotonic logics that have been developed over the years (e.g., Reiter's default logic [63], logic programming under the answer-set [25] or well-founded [26] semantics, defeasible logic [58]) can be viewed at least in part as attempts to resolve conflicts between assertions. E.g., the extended logic program

$$
\begin{aligned}
p &:- \ \sim \neg p\\
\neg p &:- \ \sim p\\
q &:- p\\
q &:- \neg p
\end{aligned}
$$

possesses two answer sets ($\{p, q\}$, and $\{\neg p, q\}$), and q is considered a consequence of the program. The first two rules can be seen as asserting presumptive (and conflicting) evidence for both p and

Table 10
Test knowledge bases

Name	Description
Amino Acid	An $\mathcal{ALCF}(D)$ ontology pertaining to amino-acids. `http://www.co-ode.org/ontologies/amino-acid/2006/05/18/amino-acid.owl`
Pizza	A $\mathcal{SHOIN}$ ontology making frequent use of ∃- and ∀-restrictions, and infrequent use of nominals. `http://www.co-ode.org/ontologies/pizza/2007/02/12/`
Proton	A $\mathcal{SHI}(D)$ ontology using concepts from the PROTON ontologies, with disjointness axioms added. `http://proton.semanticweb.org/`
TAMBIS	A $\mathcal{SHIN}$ ontology for the molecular biology domain. `http://owl.cs.manchester.ac.uk/repository/` (`tambis-patched.owl`)
Wine	A $\mathcal{SHOIN}(D)$ ontology making heavy use of nominals. `http://www.w3.org/TR/owl-guide/wine.rdf`
University 1	A $\mathcal{ALEHI}+(D)$ ontology made using the Lehigh Ontology Benchmark Suite [27]. DisjointClasses axioms were manually added. A separate inconsistent version was created by adding SameAs assertions for individuals in disjoint classes. `http://swat.cse.lehigh.edu/projects/lubm/`
University 2	A larger version of University 1. An inconsistent version of the ontology was also generated.

Table 11
Knowledge base sizes

KB	Concepts	Ind.	GCIs	Concept Assert.	(Obj) Role Assert.	Disjoint Concepts	Equiv Concepts
Amino Acid	46	0	238	0	0	199	12
Pizza	100	5	259	10	0	398	15
Proton	266	0	278	0	0	1297	0
TAMBIS	395	0	343	0	0	21	150
Wine	77	161	126	182	246	1	61
University 1	43	2640	36	2840	11233	6	6
University 2	43	13865	36	14665	41095	6	6

Table 12
Size increase (relative to original)

	Concept			GCIs		
Ontology	Internal	Strong	Material	Internal	Strong	Material
Amino Acid	1.7	2	1.98	1	1.84	1
Pizza	1.77	2.03	1.75	1	1.27	1
Proton	1.99	2	1.39	1	1.1	1
TAMBIS	1.07	2	1.72	1	1.94	1
Wine	1.03	2.82	2.53	1	1.99	1
University 1	1.51	2	1.72	1	1.43	1
University 2	1.51	2	1.72	1	1.43	1

Disjoint class axioms and equivalent class axioms
were interpreted as sets of inclusion axioms.

Table 13
Average classification times, in milliseconds.

Knowledge Base	**Original**	**Internal**	**Strong**	**Material**
Amino Acid	102	236	`time-limit`	18800
Pizza	344	364	`time-limit`	`mem-limit`
Proton	121	592	739	302850
TAMBIS	337	1098	`mem-limit`	`mem-limit`
Wine	23105	84201	`time-limit`	`time-limit`
University 1	251	4415	`time-limit`	`mem-limit`
University 1 (Inconsistent)	N/A	4838	`time-limit`	`mem-limit`
University 2	1066	58542	`time-limit`	`mem-limit`
University 2 (Inconsistent)	N/A	58805	`time-limit`	`mem-limit`

Computer: AMD Athlon II X2, 6GB DDR2 RAM, Java SE 6 (64-bit, 4GB RAM).
`time-limit`: program was manually terminated after 15 minutes.
`mem-limit`: program exceeded available memory (and automatically terminated).

$\neg p$, and the conflict is resolved by disallowing both p and $\neg p$ from appearing in a common answer set. The answer-set semantics, like Reiter's default logic and the other common semantics for logic programs, espouses the principle of explosion. The mechanism by which conflict is handled is default negation, and there are programs that are inherently inconsistent (regardless of whether default negation appears); drawing reasonable conclusions from such programs is impossible under the most common semantics for them. Nevertheless, alternative paraconsistent semantics for logic programs [19,20,71,72], and for default logic [29,73,74], have been proposed.

The most obvious distinction between such formalisms and the paraconsistent logics presented here is that the former are nonmonotonic whereas the latter (like the underlying description logics upon which they are based) are monotonic. While nonmonotonicity is justifiable in many cases (arguably, it is required if one is to model certain forms of reasoning), it typically comes with a computational penalty. Unrestricted default logic is not recursively enumerable, for instance. Furthermore, it would be difficult to embed an inherently nonmonotonic logic into a monotonic one, which is what we have done here with the paraconsistent description logics.

Even within the paraconsistent reasoning community, there are approaches for achieving paraconsistency that differ from the multivalued approach we have described here. Jaśkowski's [23,38] *discussive* or *discursive* logics (or "logics of discussion") are considered to the first paraconsistent logic in modern times. In such logics, a proposition may be viewed as true whenever it has been proposed by a participant in a discussion. Since it is possible for participants to disagree, statements can be both true and false. The semantics for discussive logics are typically specified indirectly by a modal logic (e.g., S5). A sentence $\mathcal{P}$ of the theory, called a *discussive assertion*, would be treated as $\Diamond\mathcal{P}$ ("possibly $\mathcal{P}$") and interpreted according to the modal logic. As in the multivalued logics presented here, many common rules of inference do not naturally hold under this interpretation. In particular, Modus Ponens fails, as does the Rule of Adjunction ($\{\mathcal{P}, \mathcal{Q}\}/ \therefore \mathcal{P} \wedge \mathcal{Q}$). Jaśkowski thus developed both a "discussive implication" and a "discussive conjunction".

Newton da Costa was the first to develop and publish first-order paraconsistent logics in the 1950s and 1960s, and for that reason he is often considered the father of paraconsistency [16,18]. Jaśkowski's work was confined to the propositional level and was not widely known at the time [17]. Da Costa is best known for his $\mathcal{C}$-systems, which constitute an infinite hierarchy of paraconsistent logics. Their defining characteristic is that consistency is explicitly represented in the object language itself. I.e., a new unary operator $\circ$ is added, with $\circ\mathcal{P}$ stating that $\mathcal{P}$ is consistent. The logics are typically defined axiomatically, avoiding explosion by replacing certain classical axioms with weakened counterparts. Semantically, the logics break the truth-functional connection between a sentence and its negation. Much later, Carnielli, Coniglio, and Marcos would generalize the $\mathcal{C}$-systems, defining the *logics of formal inconsistency* [14,15].

Neither discussive logic nor the logics based on da Costa's have in our opinion received much coverage in fields related to computer science. Within the philosophical and mathematical communities, discussive logic has received quite a bit of criticism, particularly for the failure of Adjunction. In our view, however, many critics appear simply to be unaware of Jaśkowski's work on discussive conjunction. Both sorts of logic have received criticism for breaking the usual behavior of negation, conjunction, and disjunction (e.g., in at least some of $\mathcal{C}$-systems, De Morgan's laws do not hold). One of the few frameworks utilizing a logic of formal inconsistency in an information system is described in [21]. There, ideas from the stable model and well-founded semantics for logic programs are combined to create a paraconsistent version of Datalog.

The formalisms we have noted above allow an assertion and its negation to be simultaneously true (in some sense), and the formalisms permit reasoning under these conditions. In a way, such systems can be viewed as tolerating inconsistency. There are indeed a great many other approaches to handling inconsistency (see, e.g., the papers found in [11]). In one common view, inconsistencies in a knowledge base are taken as errors, and algorithms are developed to identify the errors and repair the knowledge base. Classic examples of this would be Reiter's work [64] on fault diagnosis and the wide body of literature devoted to belief revision and truth maintenance systems [1,22,24]. Other work focuses on the identification of answers to queries that are consistent with repaired versions of inconsistent knowledge bases (regardless of whether or not the knowledge base actually has been repaired) [2,13]. We note that the view underlying such work—i.e., that inconsistencies are faults or errors in the system—is often correct, and in such cases it is important to identify the sources of the error. This is something that the multivalued approach described here does not do. In the semantic web context, work on ontology debugging and determining justifications in ontologies (e.g., [40,75]) may also be considered relevant.

Returning to multivalued logics, the logics presented in this paper are related to earlier 4-valued description logics developed by Patel-Schneider [59,60] and also Straccia, Sebastiani, and Meghini [54,55,68,69]. $\mathcal{SROIQ}4$ is similar to these in the sense that all are 4-valued and essentially based on the logics of Belnap, but the earlier logics correspond neither syntactically nor semantically to $\mathcal{SROIQ}4$. The earlier logics are in fact syntactically closer to $\mathcal{ALC}$ and $\mathcal{SHIQ}$, but there are differences even when restricted to these. Patel-Schneider's logic does not allow concept union or existential role restrictions, and cardinality restrictions are unqualified. Equality in the logic is nonclassical—specifically, it is 4-valued, reflexive, and symmetric, but it is not necessarily transitive. Among other things, this affects the definition of cardinality restrictions—concepts no longer have a single cardinality in the logic.

Straccia et al. [54,55,68,69] provide 4-valued semantics for $\mathcal{ALC}$-like languages similar to but distinct from Patel-Schneider's, and in some cases sound and complete sequent calculi are provided [55,69]. The interpretations for existential and universal role restrictions are the same as those used in $\mathcal{SROIQ}4$ (from a syntactic and semantic standpoint, the $\mathcal{ALC}$ fragment of $\mathcal{SROIQ}4$ appears to identical to the logic described in [68]), which is distinct from those of Patel-Schneider. Subsumption in the logics is what we have called *weak* subsumption: C is subsumed by D *iff* $p^+(C^\mathcal{I}) \subseteq p^+(D^\mathcal{I})$ in every 4-interpretation. Similarly, equivalence is the weak equivalence defined earlier. Since cardinality restrictions are not used in the logics, equality is not addressed.

The work here on $\mathcal{SROIQ}4$ and related logics constitutes an extended version of materials appearing in both [46] and [53] (neither ELP nor the empirical tests are discussed in those works, however), which are in turn based on earlier work [47,48,49,50,51]. Regarding this earlier work, the logic $\mathcal{ALC}4$ is discussed in [47,48,49]. Embedding $\mathcal{ALC}4$ into classical $\mathcal{ALC}$ is discussed in [47] and [48], while resolution-based reasoning algorithms are presented in [48] and [49]. $\mathcal{SHOIN}4$ and its embedding into classical $\mathcal{SHOIN}$ [35] are presented in [51]. In $\mathcal{SHOIN}4$, unqualified cardinality restrictions are allowed, as are nominals (which are treated classically). $\mathcal{SHIQ}4$ is presented in [50], as are paraconsistent versions of $\mathcal{EL}^{++}$, Horn-DLs, and DL-Lite logics. The latter logics are analyzed more fully here and in [46]. With one exception [49], each of these papers describes embedding a paraconsistent logic into a classical counterpart.

We are unaware of anything similar to this in the work by Patel-Schneider and Straccia et al.

Lang [45] has defined paraconsistent semantics for $\mathcal{ALC}$ and $\mathcal{SHIQ}$ (which, incidentally, he also calls $\mathcal{ALC}4$ and $\mathcal{SHIQ}4$) by translating them into a paraconsistent first-order logic based on Arieli and Avron's propositional logic. As noted before, the 4-valued ELP semantics presented here is similar to his first-order logic semantics. Lang also provides separate schemes for translating his logic into FOL and for translating paraconsistent $\mathcal{ALC}4$ and $\mathcal{SHIQ}4$ directly into their classical counterparts. Paraconsistent equality and inequality are introduced as special binary relations, together with suitable axioms. An implication operator, which we refer to using $\rightarrow_L$, is also defined.

$\rightarrow_L$	n	0	1	b
n	1	1	1	1
0	1	b	1	b
1	n	n	1	1
b	n	0	1	b

Since $\{\neg(\mathcal{P} \rightarrow_L \mathcal{Q})\} \models_4 (\neg\mathcal{P} \wedge \neg\mathcal{Q})$, the connective is classically unsound.

More recently, Kamide [41] describes a 4-valued paraconsistent variant of $\mathcal{ALC}$ making use of both classical and paraconsistent negation. An embedding theorem similar to the ones described here is also given, as is a sound and complete tableau algorithm. Without the additional negation, the logic appears to be the same as $\mathcal{ALC}4$ (though there are certain notational differences). The use of the additional negation strengthens the logic considerably, however. Indeed, it appears that all classical $\mathcal{ALC}$ consequences are regained.

Zhang et al. [70] present a paraconsistent variation of $\mathcal{SHIQ}$ based on Besnard and Hunter's *quasi-classical logic* [37]. It is claimed that the logic (QC-$\mathcal{SHIQ}$) remedies problems inherent to the 4-valued approach described here. Specifically, a *weak* semantics (virtually identical to $\mathcal{SHIQ}4$) is defined in which inclusion is based on internal inclusion. For the weak semantics, Modus Ponens, Modus Tollens, and Disjunctive Syllogism all fail. A *strong* semantics, building upon the weak, is provided which does not suffer from this problem (all three hold).

While it is certainly true that Disjunctive Syllogism fails for $\mathcal{SROIQ}4$ regardless of the inclusion operator used, Modus Ponens is satisfied by both internal and strong inclusion, and Modus Tollens is satisfied by strong inclusion. This is the primary point of introducing the operators in the paraconsistent logics described here (and, we imagine, in the propositional logics upon which they are based).

The failure of Disjunctive Syllogism is indeed a substantial drawback. The paraconsistent logics described here simply do not permit it under any circumstances. Arieli has recently noted [4] that Belnap's 4-valued logic is strictly weaker than quasi-classical logic.[10] Nevertheless, the main virtue of the paraconsistent DL framework presented here is that the logics can be embedded into classical formalisms. This in turn allows classical tools to be used to reason over even inconsistent knowledge bases. To our knowledge, no similar translation scheme has been created for QC-$\mathcal{SHIQ}$.

Appendix

A. Proofs

A.1. Properties of Implication Operators

Proof of Proposition 2. If $\leadsto$ satisfies Modus Ponens and the Deduction Theorem, then $\leadsto$ satisfies none of Modus Tollens, Strong Equivalence, or Transposition.

Proof. Suppose $\leadsto$ satisfies Modus Ponens and the Deduction Theorem. Let A and B be atomic, $\mathcal{P} = (A \vee B)$, $\mathcal{Q} = (A \wedge \neg A)$, and $K = \{\mathcal{Q}\}$. Let v be a truth-value assignment such that $v(A) = b$ and $v(B) = 1$. Observe that $v(\mathcal{P}) = 1$ and $v(\mathcal{Q}) = b$, and so every element of K is designated on v. We consider Modus Tollens, Strong Equivalence and Transposition separately.

1. Given the definitions of $\mathcal{Q}$ and K above, $K \cup \{\mathcal{P}\} \models_4 \mathcal{Q}$ and $K \models_4 \neg\mathcal{Q}$. Since $\leadsto$ satisfies the Deduction Theorem, $K \models_4 \mathcal{P} \leadsto \mathcal{Q}$. K is designated on v and yet $v(\neg\mathcal{P}) = 0$, and so $K \not\models_4 \neg\mathcal{P}$. As such, $\leadsto$ does not satisfy Modus Tollens.

[10] The comparison was made using the simple propositional language defined over $\neg$, $\wedge$, and $\vee$. Arieli also points out that quasi-classical logic is in some cases *too strong*, in that Disjunctive Syllogism is permitted in too many cases. E.g., $\{A, \neg A, A \vee B\} \vdash_{QC} B$, which will strike many as counter-intuitive.

2. Given the definitions of $\mathcal{P}$, $\mathcal{Q}$, and K above, it follows that $K \models_4 \mathcal{P} \rightsquigarrow \mathcal{Q}$ and $K \models_4 \mathcal{Q} \rightsquigarrow \mathcal{P}$. K is designated on v, and so both $(\mathcal{P} \rightsquigarrow \mathcal{Q})$ and $(\mathcal{Q} \rightsquigarrow \mathcal{P})$ are designated on v. Yet $v(\mathcal{P}) = 1$ and $v(\mathcal{Q}) = b$, and so Strong Equivalence is not satisfied.
3. As in 1 above, $K \models_4 \mathcal{P} \rightsquigarrow \mathcal{Q}$ and $K \models_4 \neg\mathcal{Q}$. If $K \models_4 \neg\mathcal{Q} \rightsquigarrow \neg\mathcal{P}$, then by Modus Ponens, $K \models_4 \neg\mathcal{P}$. However, K is designated on v but $v(\neg\mathcal{P}) = 0$, and so it can't be that $K \models_4 \neg\mathcal{Q} \rightsquigarrow \neg\mathcal{P}$. Consequently, Transposition is not satisfied.

$\square$

Observe that it does not matter whether $\rightsquigarrow$ is Supraclassical or satisfies Identity. Furthermore, only in the counterexample for Transposition is Modus Ponens actually used.

A.2. $\mathcal{SROIQ}4$

Proof of Proposition 17. For any $\mathcal{SROIQ}4$ concepts C, D and 4-interpretation $\mathcal{I}$, the following hold:

1. $(\neg\top)^\mathcal{I} = \bot^\mathcal{I}$
2. $(\neg\bot)^\mathcal{I} = \top^\mathcal{I}$
3. $(\neg\neg C)^\mathcal{I} = C^\mathcal{I}$
4. $(\neg(C \sqcap D))^\mathcal{I} = (\neg C \sqcup \neg D)^\mathcal{I}$
5. $(\neg(C \sqcup D))^\mathcal{I} = (\neg C \sqcap \neg D)^\mathcal{I}$
6. $(\neg\exists R.C)^\mathcal{I} = (\forall R.\neg C)^\mathcal{I}$
7. $(\neg\forall R.C)^\mathcal{I} = (\exists R.\neg C)^\mathcal{I}$
8. $(\neg \leq nR.C)^\mathcal{I} = (\geq n+1R.C)^\mathcal{I}$
9. $(\neg \geq nR.C)^\mathcal{I} = (\leq n-1R.C)^\mathcal{I}$

Proof. We show selected cases.

1.1. $p^+((\neg\top)^\mathcal{I}) = p^-(\top^\mathcal{I}) = \varnothing = p^+(\bot^\mathcal{I})$
1.2. $p^-((\neg\top)^\mathcal{I}) = p^+(\top^\mathcal{I}) = \Delta^\mathcal{I} = p^-(\bot^\mathcal{I})$
3.1. $p^+((\neg\neg C)^\mathcal{I}) = p^-((\neg C)^\mathcal{I}) = p^+(C^\mathcal{I})$.
3.2. $p^-((\neg\neg C)^\mathcal{I}) = p^+((\neg C)^\mathcal{I}) = p^-(C^\mathcal{I})$.
4.1. $p^+((\neg(C \sqcap D))^\mathcal{I}) = p^-((C \sqcap D)^\mathcal{I}) = p^-(C^\mathcal{I}) \cup p^-(D^\mathcal{I}) = p^+((\neg C)^\mathcal{I}) \cup p^+((\neg D)^\mathcal{I}) = p^+((\neg C \sqcup \neg D)^\mathcal{I})$
4.2. $p^-((\neg(C \sqcap D))^\mathcal{I}) = p^+((C \sqcap D)^\mathcal{I}) = p^+(C^\mathcal{I}) \cap p^+(D^\mathcal{I}) = p^-((\neg C)^\mathcal{I}) \cap p^-((\neg D)^\mathcal{I}) = p^-((\neg C \sqcup \neg D)^\mathcal{I})$
6.1. $p^+((\neg\exists R.C)^\mathcal{I}) = p^-((\exists R.C)^\mathcal{I}) = \{x|(\forall y)[(x,y) \in p^+(R^\mathcal{I}) \mapsto y \in p^-(C^\mathcal{I})]\} = \{x|(\forall y)[(x,y) \in p^+(R^\mathcal{I}) \mapsto y \in p^+((\neg C)^\mathcal{I})]\} = p^+((\forall R.\neg C)^\mathcal{I})$
6.2. $p^-((\neg\exists R.C)^\mathcal{I}) = p^+((\exists R.C)^\mathcal{I}) = \{x|(\exists y)[(x,y) \in p^+(R^\mathcal{I}) \wedge y \in p^+(C^\mathcal{I})]\} = \{x|(\exists y)[(x,y) \in p^+(R^\mathcal{I}) \wedge y \in p^-((\neg C)^\mathcal{I})]\} = p^-((\forall R.\neg C)^\mathcal{I})$
8.1. $p^+((\neg \leq nR.C)^\mathcal{I}) = p^-((\leq nR.C)^\mathcal{I}) = \{x|\sharp\{y|(x,y) \in p^+(R^\mathcal{I}) \wedge y \in p^+(C^\mathcal{I})\} > n\} = \{x|\sharp\{y|(x,y) \in p^+(R^\mathcal{I}) \wedge y \in p^+(C^\mathcal{I})\} \geq (n+1)\} = p^+((\geq (n+1)R.C)^\mathcal{I})$
8.2. $p^-((\neg \leq nR.C)^\mathcal{I}) = p^+((\leq nR.C)^\mathcal{I}) = \{x|\sharp\{y|(x,y) \in p^+(R^\mathcal{I}) \wedge y \notin p^-(C^\mathcal{I})\} \leq n\} = \{x|\sharp\{y|(x,y) \in p^+(R^\mathcal{I}) \wedge y \notin p^-(C^\mathcal{I})\} < (n+1)\} = p^-((\geq (n+1)R.C)^\mathcal{I})$

$\square$

Proof of Proposition 20. Let C and D be $\mathcal{SROIQ}4$ concepts.

1. C is weakly subsumed by D *iff* $C \sqsubset D$ is satisfied on each $\mathcal{I}$.
2. C is strongly subsumed by D *iff* $C \to D$ is satisfied on each $\mathcal{I}$.
3. C and D are weakly equivalent iff $C \sqsubset D$ and $D \sqsubset C$ are both satisfied on each $\mathcal{I}$.
4. C and D are strongly equivalent iff $C \to D$ and $D \to C$ are both satisfied on each $\mathcal{I}$.

Proof.

1. This follows directly from the definition of weak subsumption and satisfaction $C \sqsubset D$.
2. Again, this follows by definition of strong subsumption and satisfaction $C \to D$.
3. If $(C \leftrightarrow D)$, then for any $\mathcal{I}$, $C^\mathcal{I} = D^\mathcal{I}$, and so $p^+(C^\mathcal{I}) \subseteq p^+(D^\mathcal{I})$ and $p^-(D^\mathcal{I}) \subseteq p^-(C^\mathcal{I})$, and $p^+(D^\mathcal{I}) \subseteq p^+(C^\mathcal{I})$ and $p^-(C^\mathcal{I}) \subseteq p^-(D^\mathcal{I})$. Both $(C \to D)$ and $(D \to C)$ are satisfied by $\mathcal{I}$. If $(C \to D)$ and $(D \to C)$ are satisfied by $\mathcal{I}$, then by definition, $p^+(C^\mathcal{I}) \subseteq p^+(D^\mathcal{I})$ and $p^-(D^\mathcal{I}) \subseteq p^-(C^\mathcal{I})$, and $p^+(D^\mathcal{I}) \subseteq p^+(C^\mathcal{I})$ and $p^-(C^\mathcal{I}) \subseteq p^-(D^\mathcal{I})$. Hence, $C^\mathcal{I} = D^\mathcal{I}$. Generalizing on $\mathcal{I}$, $(C \leftrightarrow D)$.
4. Clearly, $p^+(C^\mathcal{I}) = p^+(D^\mathcal{I})$ *iff* $p^+(C^\mathcal{I}) \subseteq p^+(D^\mathcal{I})$ and $p^+(D^\mathcal{I}) \subseteq p^+(C^\mathcal{I})$.

$\square$

Below, Δ refers to the domain of discourse shared by both $\mathcal{I}_4$ and $\mathcal{I}_2$.

Proof of Proposition 21. If $\mathcal{I}_2$ is a 2-interpretation and $\mathcal{I}_4$ its 4-counterpart, then for any concept C

$$C^{\mathcal{I}_4} = \langle C^{\mathcal{I}_2}, \Delta - C^{\mathcal{I}_2} \rangle.$$

Proof. We induct on the degree of C. If $C \in N_C \cup N_o$ or $C = \exists R.Self$, the claim holds by definition. For $\top$, $p^+(\top^{\mathcal{I}_4}) = \Delta = \top^{\mathcal{I}_2}$ and $p^-(\top^{\mathcal{I}_4}) = \varnothing = \Delta - \Delta = \Delta - \top^{\mathcal{I}_2}$. For bottom, $p^+(\bot^{\mathcal{I}_4}) = \varnothing = \bot^{\mathcal{I}_2}$ and $p^-(\bot^{\mathcal{I}_4}) = \Delta = \Delta - \varnothing = \Delta - \bot^{\mathcal{I}_2}$. For the inductive cases, suppose the claim holds for all concepts of degree $< n$ and that C has degree n. We consider the cases where C is one of $\neg D$, $(D \sqcap E)$, $(\exists R.D)$, or $(\geq nR.D)$. The remaining cases are similar.

- $(\neg \mathbf{D})^{\mathcal{I}_4} = \langle p^-(D^{\mathcal{I}_4}), p^+(D^{\mathcal{I}_4})\rangle = \langle \Delta - D^{\mathcal{I}_2}, D^{\mathcal{I}_2})\rangle = \langle (\neg D)^{\mathcal{I}_2}, \Delta - (\neg D)^{\mathcal{I}_2})\rangle$.
- $(\mathbf{D} \sqcap \mathbf{E})^{\mathcal{I}_4} = \langle (p^+(D^{\mathcal{I}_4}) \cap p^+(E^{\mathcal{I}_4})), (p^-(D^{\mathcal{I}_4}) \cup p^-(E^{\mathcal{I}_4}))\rangle = \langle (D^{\mathcal{I}_2} \cap E^{\mathcal{I}_2}), ((\Delta - D^{\mathcal{I}_2}) \cup (\Delta - E^{\mathcal{I}_2}))\rangle = \langle (D^{\mathcal{I}_2} \cap E^{\mathcal{I}_2}), (\Delta - (D^{\mathcal{I}_2} \cap E^{\mathcal{I}_2}))\rangle$.
- $(\exists \mathbf{R}.\mathbf{D})^{\mathcal{I}_4} = \langle P, N\rangle$, where $P = \{x | (\exists y)[(x,y) \in p^+(R^{\mathcal{I}_4}) \wedge y \in p^+(D^{\mathcal{I}_4})]\} = \{x | (\exists y)[(x,y) \in R^{\mathcal{I}_2} \wedge y \in D^{\mathcal{I}_2}]\} = (\exists R.D)^{\mathcal{I}_2}$; and $N = \{x | (\forall y)[(x,y) \in p^+(R^{\mathcal{I}_4}) \mapsto y \in p^-(D^{\mathcal{I}_4})]\} = \{x | (\forall y)[(x,y) \in R^{\mathcal{I}_2} \mapsto y \in (\Delta - D^{\mathcal{I}_2})]\} = \{x | (\forall y)[(x,y) \in R^{\mathcal{I}_2} \mapsto y \notin D^{\mathcal{I}_2}]\} = \{x | \neg\neg(\forall y)[(x,y) \in R^{\mathcal{I}_2} \mapsto y \notin D^{\mathcal{I}_2}]\} = \{x | \neg(\exists y)[(x,y) \in R^{\mathcal{I}_2} \wedge y \in D^{\mathcal{I}_2}]\} = \Delta - (\exists R.D)^{\mathcal{I}_2}$.
- $(\geq \mathbf{nR}.\mathbf{D})^{\mathcal{I}_4} = \langle P, N\rangle$, where $P = \{x | \sharp\{y | (x,y) \in p^+(R^{\mathcal{I}_4}) \wedge y \in p^+(D^{\mathcal{I}_4})\} \geq n\} = \{x | \sharp\{y | (x,y) \in R^{\mathcal{I}_2} \wedge y \in D^{\mathcal{I}_2}\} \geq n\} = D^{\mathcal{I}_2}$; and $N = \{x | \sharp\{y | (x,y) \in p^+(R^{\mathcal{I}_4}) \wedge y \notin p^-(D^{\mathcal{I}_4})\} < n\} = \{x | \sharp\{y | (x,y) \in p^+(R^{\mathcal{I}_4}) \wedge y \in p^+(D^{\mathcal{I}_4})\} < n\} = \{x | \sharp\{y | (x,y) \in R^{\mathcal{I}_2} \wedge y \in D^{\mathcal{I}_2}\} < n\} = \Delta - (\geq nR.D)^{\mathcal{I}_2}$.

□

The below two lemmas are needed to prove Proposition 22, which relates the 2-models of an axiom A to its 4-models.

Lemma 61. *Let $\mathcal{I}_2$ be a 2-interpretation and $\mathcal{I}_4$ its 4-counterpart. If $R \in N_R$, then $R^{-\mathcal{I}_4} = \langle R^{-\mathcal{I}_2}, \Delta^2 - R^{-\mathcal{I}_2}\rangle$.*

Proof. By definition of $\mathcal{I}_4$, $R^{\mathcal{I}_4} = \langle R^{\mathcal{I}_2}, \Delta^2 - R^{\mathcal{I}_2}\rangle$. From this, $R^{-\mathcal{I}_4} = \langle R^{-\mathcal{I}_2}, (\Delta^2 - R^{\mathcal{I}_2})^-\rangle$. Observe that $(\Delta^2 - R^{\mathcal{I}_2})^- = (\Delta^2)^- - (R^{\mathcal{I}_2})^- = \Delta^2 - R^{-\mathcal{I}_2}$. And so $R^{-\mathcal{I}_4} = \langle R^{-\mathcal{I}_2}, \Delta^2 - R^{-\mathcal{I}_2})\rangle$. □

Lemma 62. *If $\mathcal{I}_2$ is a 2-interpretation and $\mathcal{I}_4$ its 4-counterpart, then for roles $R_1, \ldots, R_n$, $(R_1 \circ \ldots \circ R_n)^{\mathcal{I}_4} = \langle (R_1 \circ \ldots \circ R_n)^{\mathcal{I}_2}, \Delta^2 - (R_1 \circ \ldots \circ R_n)^{\mathcal{I}_2}\rangle$.*

Proof. That $p^+((R_1 \circ \ldots \circ R_n)^{\mathcal{I}_4}) = (R_1 \circ \ldots \circ R_n)^{\mathcal{I}_2}$ and $p^-((R_1 \circ \ldots \circ R_n)^{\mathcal{I}_4}) = \Delta^2 - (R_1 \circ \ldots \circ R_n)^{\mathcal{I}_2}$ follows by definition of $\mathcal{I}_4$ for roles and Lemma 61. □

Proof of Proposition 22. If $\mathcal{I}_2$ is a 2-interpretation and $\mathcal{I}_4$ its 4-counterpart, then for any axiom A, $\mathcal{I}_4$ is a 4-model of A *iff* $\mathcal{I}_2$ is a 2-model of A.

Proof. (By cases):

- $\mathbf{A} = \mathbf{C}(\mathbf{a})$: $\mathcal{I}_4$ is a 4-model of $C(a)$ *iff* $a^{\mathcal{I}_4} \in p^+(C^{\mathcal{I}_4})$ *iff* $a^{\mathcal{I}_2} \in C^{\mathcal{I}_2}$ *iff* $\mathcal{I}_2$ is a 2-model of $C(a)$.
- $\mathbf{A} = \mathbf{R}(\mathbf{a}, \mathbf{b})$: $\mathcal{I}_4$ is a 4-model of $R(a,b)$ *iff* $(a^{\mathcal{I}_4}, b^{\mathcal{I}_4}) \in p^+(R^{\mathcal{I}_4})$ *iff* $(a^{\mathcal{I}_2}, b^{\mathcal{I}_2}) \in R^{\mathcal{I}_2}$ *iff* $\mathcal{I}_2$ is a 2-model of $R(a,b)$.
- $\mathbf{A} = \neg\mathbf{R}(\mathbf{a}, \mathbf{b})$: $\mathcal{I}_4$ is a 4-model of $\neg R(a,b)$ *iff* $(a^{\mathcal{I}_4}, b^{\mathcal{I}_4}) \in p^-(R^{\mathcal{I}_4})$ *iff* $(a^{\mathcal{I}_2}, b^{\mathcal{I}_2}) \notin R^{\mathcal{I}_2}$ *iff* $\mathcal{I}_2$ is a 2-model of $\neg R(a,b)$.
- $\mathbf{A} = \mathbf{C} \sqsubset \mathbf{D}$: $\mathcal{I}_4$ is a 4-model of $(C \sqsubset D)$ *iff* $p^+(C^{\mathcal{I}_4}) \subseteq p^+(D^{\mathcal{I}_4})$ *iff* $C^{\mathcal{I}_2} \subseteq D^{\mathcal{I}_2}$ *iff* $\mathcal{I}_2$ is a 2-model of $C \sqsubset D$.
- $\mathbf{A} = \mathbf{C} \mapsto \mathbf{D}$: $\mathcal{I}_4$ is a 4-model of $(C \mapsto D)$ *iff* $\Delta - p^-(C^{\mathcal{I}_4}) \subseteq p^+(D^{\mathcal{I}_4})$ *iff* $p^+(C^{\mathcal{I}_4}) \subseteq p^+(D^{\mathcal{I}_4})$ *iff* $C^{\mathcal{I}_2} \subseteq D^{\mathcal{I}_2}$ *iff* $\mathcal{I}_2$ is a 2-model of $C \mapsto D$.
- $\mathbf{A} = \mathbf{C} \to \mathbf{D}$: $\mathcal{I}_4$ is a 4-model of $(C \to D)$ *iff* $(p^+(C^{\mathcal{I}_4}) \subseteq p^+(D^{\mathcal{I}_4})$ and $p^-(D^{\mathcal{I}_4}) \subseteq p^-(C^{\mathcal{I}_4}))$ *iff* $(C^{\mathcal{I}_2} \subseteq D^{\mathcal{I}_2}$ and $\Delta - p^-(C^{\mathcal{I}_4}) \subseteq \Delta - p^-(D^{\mathcal{I}_4}))$ *iff* $(C^{\mathcal{I}_2} \subseteq D^{\mathcal{I}_2}$ and $C^{\mathcal{I}_2} \subseteq D^{\mathcal{I}_2})$ *iff* $\mathcal{I}_2$ is a 2-model of $C \to D$.
- $\mathbf{A} = \mathbf{R_1} \circ \ldots \circ \mathbf{R_n} \sqsubset \mathbf{R_{n+1}}$: $\mathcal{I}_4$ is a 4-model of $R_1 \circ \ldots \circ R_n \sqsubset R_{n+1}$ *iff* $p^+((R_1 \circ \ldots \circ R_n)^{\mathcal{I}_4}) \subseteq p^+(R_{n+1}^{\mathcal{I}_4})$ *iff* $(R_1 \circ \ldots \circ R_n)^{\mathcal{I}_2} \subseteq R_{n+1}^{\mathcal{I}_2}$ *iff* $\mathcal{I}_2$ is a 2-model of $R_1 \circ \ldots \circ R_n \sqsubset R_{n+1}$.
- $\mathbf{A} = \mathbf{Ref}(\mathbf{R})$: $\mathcal{I}_4$ is a 4-model of $Ref(R)$ *iff* $\{(x,x) | x \in \Delta\} \subseteq p^+((R^{\mathcal{I}_4}))$ *iff* $\{(x,x) | x \in \Delta\} \subseteq R^{\mathcal{I}_2}$ *iff* $\mathcal{I}_2$ is a 2-model of $Ref(R)$.
- $\mathbf{A} = \mathbf{Irr}(\mathbf{R})$: $\mathcal{I}_4$ is a 4-model of $Irr(R)$ *iff* $\{(x,x) | x \in \Delta\} \subseteq p^-((R^{\mathcal{I}_4}))$ *iff* $\{(x,x) | x \in \Delta\} \subseteq (\Delta \times \Delta) - R^{\mathcal{I}_2}$ *iff* $\{(x,x) | x \in \Delta\} \cap R^{\mathcal{I}_2} = \varnothing$ *iff* $\mathcal{I}_2$ is a 2-model of $Irr(R)$.
- $\mathbf{A} = \mathbf{Dis}(\mathbf{R}, \mathbf{S})$: $\mathcal{I}_4$ is a 4-model of $Dis(R,S)$ *iff* $(p^+(R^{\mathcal{I}_4}) \subseteq p^-(S^{\mathcal{I}_4})$ and $p^+(S^{\mathcal{I}_4}) \subseteq p^-(R^{\mathcal{I}_4}))$ *iff* $(R^{\mathcal{I}_2} \subseteq \Delta^2 - S^{\mathcal{I}_2}$ and $S^{\mathcal{I}_2} \subseteq \Delta^2 - R^{\mathcal{I}_2})$ *iff* $R^{\mathcal{I}_2} \cap S^{\mathcal{I}_2} = \varnothing$ *iff* $\mathcal{I}_2$ is a 2-model of $Dis(R,S)$.

Since (in)equality assertions have the same semantics in both 2-interpretations and 4-interpretations, we need show nothing for them.

□

A.3. Removing Gaps and Gluts

Proof of Proposition 30. A 4-valued interpretation $\mathcal{I}$ of $EM(KB)$ is a 4-model of $EM(KB)$ *iff* for each concept C of KB, $p^+(C^\mathcal{I}) \cup p^-(C^\mathcal{I}) = \Delta^\mathcal{I}$.

Proof. **(LR)** Let $\mathcal{I}$ 4-model $EM(KB)$. We induct on the degree of the concept C. For $C = \top$ and $C = \bot$, the claim holds by definition of $\top$ and $\bot$. If $C \in N_C \cup N_o$ or has the form $\exists R.Self$, since $\mathcal{I}$ 4-satisfies $EM(KB)$, $\Delta^\mathcal{I} \subseteq p^+(C^\mathcal{I}) \cup p^+((\neg C)^\mathcal{I})$. Since $p^+(C^\mathcal{I}) \cup p^+((\neg C)^\mathcal{I}) \subseteq \Delta^\mathcal{I}$, the claim must hold. For the inductive cases, suppose the claim holds for concepts of degree $< n$ and that C has degree n. We consider selective cases. Proofs for the cases left out are analogous.

1. $\mathbf{C} = (\mathbf{D} \sqcap \mathbf{E})$: If $d \notin p^+(C^\mathcal{I})$, then $d \notin p^+(D^\mathcal{I})$ or $d \notin p^+(E^\mathcal{I})$. By ind. hyp., $d \in p^-(D^\mathcal{I})$ or $d \in p^-(E^\mathcal{I})$, and so $d \in p^-((D \sqcap E)^\mathcal{I})$. I.e., $d \in p^-(C^\mathcal{I})$.
2. $\mathbf{C} = (\forall \mathbf{R}.\mathbf{D})$: If $d \notin p^+(C^\mathcal{I})$, then there exists a d' such that $(d, d') \in p^+(R^\mathcal{I})$ and $d' \notin p^+(D^\mathcal{I})$. But then by ind. hyp., $d' \in p^-(D^\mathcal{I})$, and so $d \in p^-(C^\mathcal{I})$.
3. $\mathbf{C} = (\leq \mathbf{nR}.\mathbf{D})$: If $d \notin p^+(C^\mathcal{I})$, then $\sharp\{y|(d,y) \in p^+(R^\mathcal{I}) \wedge y \notin p^-(D^\mathcal{I})\} > n\}$. But then by ind. hyp., $\sharp\{y|(d,y) \in p^+(R^\mathcal{I}) \wedge y \in p^+(D^\mathcal{I})\} > n\}$, and so $d \in p^-(C^\mathcal{I})$.

(RL) Suppose for each concept C, $p^+(C^\mathcal{I}) \cup p^-(C^\mathcal{I}) = \Delta^\mathcal{I}$. Then $p^+(C^\mathcal{I}) \cup p^+((\neg C)^\mathcal{I}) = \Delta^\mathcal{I}$, $\Delta \subseteq p^+(C^\mathcal{I}) \cup p^+((\neg C)^\mathcal{I})$, and so $p^+(\top^\mathcal{I}) \subseteq p^+((C \sqcup \neg C)^\mathcal{I})$. $\mathcal{I}$ is a 4-model of $\top \sqsubset (C \sqcup \neg C)$. Generalizing on C, $\mathcal{I}$ is a 4-model of $EM(KB)$. □

Proof of Proposition 31. A 4-valued interpretation $\mathcal{I}$ of $EFQ(KB)$ is a 4-model of $EFQ(KB)$ *iff* for each concept C of KB, $p^+(C^\mathcal{I}) \cap p^-(C^\mathcal{I}) = \varnothing$.

Proof. **(LR)** Suppose $\mathcal{I}$ 4-satisfies $EFQ(KB)$. We induct on the degree of C. For $C = \top$ and $C = \bot$, the claim holds by definition. If $C \in N_C \cup N_o$ or has the form $\exists R.Self$, $(C \sqcap \neg C) \sqsubset \bot$ is satisfied, and so $p^+(C^\mathcal{I}) \cap p^+(\neg C)^\mathcal{I}) \subseteq \varnothing$. However, $\varnothing \subseteq p^+(C^\mathcal{I}) \cap p^+(\neg C)^\mathcal{I})$, and so $p^+(C^\mathcal{I}) \cap p^-(C^\mathcal{I}) = \varnothing$. For the inductive cases, suppose the claim holds for concepts of degree $< n$ and that C has degree n. We again consider selected cases.

1. If $d \in p^+((\mathbf{D} \sqcup \mathbf{E})^\mathcal{I})$, then $d \in p^+(D^\mathcal{I})$ or $d \in p^+(E^\mathcal{I})$. By ind. hyp., $d \notin p^-(D^\mathcal{I})$ or $d \notin p^-(E^\mathcal{I})$, and so $d \notin p^-((D \sqcup E)^\mathcal{I})$.
2. If $d \in p^+((\exists \mathbf{R}.\mathbf{D})^\mathcal{I})$, then there is a d' such that $(d, d') \in p^+(R^\mathcal{I})$ and $d' \in p^+(D^\mathcal{I})$. However, by ind. hyp., $d' \notin p^-(D^\mathcal{I})$. And so $d \notin p^-((\exists R.D)^\mathcal{I})$.
3. If $d \in p^-((\leq \mathbf{nR}.\mathbf{D})^\mathcal{I})$, then $\sharp\{y|(d,y) \in p^+(R^\mathcal{I}) \wedge y \in p^+(D^\mathcal{I})\} > n$. By ind. hyp., $\sharp\{y|(d,y) \in p^+(R^\mathcal{I}) \wedge y \notin p^-(D^\mathcal{I})\} > n$, and so $d \notin p^+((\leq nR.D)^\mathcal{I})$.

(RL) If $p^+(C^\mathcal{I}) \cap p^-(C^\mathcal{I}) = \varnothing$, then $p^+(C^\mathcal{I}) \cap p^+((\neg C)^\mathcal{I}) = \varnothing$, $p^+((C \sqcap \neg C)^\mathcal{I}) = \varnothing$, and so $(C \sqcap \neg C) \sqsubseteq \bot$ is satisfied by $\mathcal{I}$. Generalizing, $\mathcal{I}$ 4-satisfies $EFQ(KB)$. □

Proof of Proposition 32. $\mathcal{I}_4$ 4-satisfies

$$EM(KB) \cup EFQ(KB)$$

iff for each concept C, $C^{\mathcal{I}_4} = \langle C^{\mathcal{I}_2}, \Delta - C^{\mathcal{I}_2} \rangle$.

Proof. **(LR)** Suppose $\mathcal{I}_4$ 4-satisfies $EM(KB) \cup EFQ(KB)$. Let C be a concept of KB. In virtue of Props. 30 and 31, $\Delta - p^+(C^{\mathcal{I}_4}) = p^-(C^{\mathcal{I}_4})$. Inducting on the degree of C, we show that $C^{\mathcal{I}_4} = \langle C^{\mathcal{I}_2}, \Delta - C^{\mathcal{I}_2} \rangle$. We consider two cases below, both requiring that $\Delta - p^+(C^{\mathcal{I}_4}) = p^-(C^{\mathcal{I}_4})$. Other cases are similar.

1. $p^+((\neg \mathbf{C})^{\mathcal{I}_4}) = p^-(C^{\mathcal{I}_4}) = \Delta - C^{\mathcal{I}_2} = (\neg C)^{\mathcal{I}_2}$. So, $\Delta - (\neg C)^{\mathcal{I}_2} = p^-((\neg C)^{\mathcal{I}_4})$.
2. $p^+((\leq \mathbf{nR}.\mathbf{C})^{\mathcal{I}_4}) = \{x|\sharp\{y|(x,y) \in p^+(R^{\mathcal{I}_4}) \wedge y \notin p^-(C^{\mathcal{I}_4})\} \leq n\} = \{x|\sharp\{y|(x,y) \in R^{\mathcal{I}_2} \wedge y \in p^+(C^{\mathcal{I}_4})\} \leq n\} = \{x|\sharp\{y|(x,y) \in R^{\mathcal{I}_2} \wedge y \in C^{\mathcal{I}_2}\} \leq n\} = (\leq nR.C)^{\mathcal{I}_2}$. And so $\Delta - (\leq nR.C)^{\mathcal{I}_2} = p^-((\leq nR.C)^{\mathcal{I}_4})$.

(RL) If $C^{\mathcal{I}_4} = \langle C^{\mathcal{I}_2}, \Delta - C^{\mathcal{I}_2} \rangle$, then $\Delta \subseteq (p^+(C^{\mathcal{I}_4}) \cup p^+(\neg C^{\mathcal{I}_4}))$ and $(p^+(C^\mathcal{I}) \cap p^+((\neg C)^{\mathcal{I}_4})) \subseteq \varnothing$. As such, $\mathcal{I}_4$ 4-satisfies $\top \sqsubset (C \sqcup \neg C)$ and $(C \sqcap \neg C) \sqsubset \bot$. □

Proof of Proposition 33. Let $\mathcal{I}_4$ be a 4-model of $EM(KB) \cup EFQ(KB)$. If A is a $\mathcal{SROIQ}4$ axiom of KB and not of the form $\neg R(a, b)$, $Irr(R)$, or $Dis(R, S)$, then $\mathcal{I}_2$ is a 2-model of A *iff* $\mathcal{I}_4$ is a 4-model of A.

Proof. We treat each case:

- $\mathcal{I}_4$ 4-satisfies $(\mathbf{C} \sqsubset \mathbf{D})$ (or $\mathbf{C} \sqsubseteq \mathbf{D}$) *iff* $p^+(C^{\mathcal{I}_4}) \subseteq p^+(D^{\mathcal{I}_4})$ *iff* $C^{\mathcal{I}_2} \subseteq D^{\mathcal{I}_2}$.
- $\mathcal{I}_4$ 4-satisfies $(\mathbf{C} \mapsto \mathbf{D})$ *iff* $\Delta - p^-(C^{\mathcal{I}_4}) \subseteq p^+(D^{\mathcal{I}_4})$ *iff* $p^+(C^{\mathcal{I}_4}) \subseteq p^+(D^{\mathcal{I}_4})$ *iff* $C^{\mathcal{I}_2} \subseteq D^{\mathcal{I}_2}$.

– $\mathcal{I}_4$ 4-satisfies $(\mathbf{C} \rightarrow \mathbf{D})$ *iff* $(p^+(C^{\mathcal{I}_4}) \subseteq p^+(D^{\mathcal{I}_4})$ and $p^-(D^{\mathcal{I}_4}) \subseteq p^-(C^{\mathcal{I}_4}))$ *iff* $(C^{\mathcal{I}_2} \subseteq D^{\mathcal{I}_2}$ and $\Delta - D^{\mathcal{I}_2} \subseteq \Delta - C^{\mathcal{I}_2})$ *iff* $C^{\mathcal{I}_2} \subseteq D^{\mathcal{I}_2}$.
– $\mathcal{I}_4$ 4-satisfies $\mathbf{R_1} \circ \ldots \circ \mathbf{R_n} \sqsubset \mathbf{R_{n+1}}$ *iff* $p^+((R_1 \circ \ldots \circ R_n)^{\mathcal{I}_4}) \subseteq p^+(R_{n+1}^{\mathcal{I}_4})$ *iff* $(R_1 \circ \ldots \circ R_n)^{\mathcal{I}_2} \subseteq R_{n+1}^{\mathcal{I}_2}$.
– $\mathcal{I}_4$ 4-satisfies $\mathbf{Ref}(\mathbf{R})$ *iff* $\{(x,x)|x \in \Delta\} \subseteq p^+(R^{\mathcal{I}_4})$ *iff* $\{(x,x)|x \in \Delta\} \subseteq R^{\mathcal{I}_2}$.
– $\mathcal{I}_4$ 4-satisfies $\mathbf{C}(\mathbf{a})$ *iff* $a^{\mathcal{I}_4} \in p^+(C^{\mathcal{I}_4})$ *iff* $a^{\mathcal{I}_2} \in C^{\mathcal{I}_2}$.
– $\mathcal{I}_4$ 4-satisfies $\mathbf{R}(\mathbf{a},\mathbf{b})$ *iff* $(a^{\mathcal{I}_4}, b^{\mathcal{I}_4}) \in p^+(R^{\mathcal{I}_4})$ *iff* $(a^{\mathcal{I}_2}, b^{\mathcal{I}_2}) \in R^{\mathcal{I}_2}$.

Since (in)equalities are treated the same in the logics, we need not consider them. □

Proof of Proposition 34. If KB is a $\mathcal{SROIQ}4$ knowledge base lacking axioms of the form $Irr(R)$, $Dis(R,S)$, or $\neg R(a,b)$ then $KB \cup EM(KB) \cup EFQ(KB)$ has a 4-model *iff* KB has a 2-model.

Proof. **(LR)** Let $\mathcal{I}_4$ 4-model $KB \cup EM(KB) \cup EFQ(KB)$. Prop. 33 applies, and so $\mathcal{I}_2$ (as described above) 2-satisfies KB. **(RL)** Let $\mathcal{I}_2$ be any 2-model of KB. We may define a 4-counterpart $\mathcal{I}_4$ as in Section 3. By Proposition 22, for any $\mathcal{SROIQ}$ axiom A of KB, $\mathcal{I}_2$ 2-satisfies A *iff* $\mathcal{I}_4$ 4-satisfies A. And so $\mathcal{I}_4$ 4-satisfies KB. Clearly, in any 2-interpretation, every axiom of $EM(KB)$ and $EFQ(KB)$ is satisfied. Again by Proposition 22, $\mathcal{I}_4$ 4-satisfies $EM(KB)$, $EFQ(KB)$. □

A.4. From $\mathcal{SROIQ}4$ to $\mathcal{SROIQ}$

Proposition 37. For any 4-interpretation $\mathcal{I}$, primed counterpart $\mathcal{I}'$, and $\mathcal{SROIQ}4$ concept C,
1. $p^+((C)^{\mathcal{I}}) = \pi(C)^{\mathcal{I}'}$
2. $p^-((C)^{\mathcal{I}}) = \pi(\neg C)^{\mathcal{I}'}$

Proof. We induct on the degree of C. If C is atomic, then $\pi(C) = C$, and by definition $p^+(C^{\mathcal{I}}) = C^{\mathcal{I}'} = \pi(C)^{\mathcal{I}'}$ and $p^-(C^{\mathcal{I}}) = C'^{\mathcal{I}'} = \pi(\neg C)^{\mathcal{I}'}$. If $C = \exists R.Self$, then $\pi(\exists R.Self) = \exists R.Self$, and by definition $(\exists R.Self)^{\mathcal{I}'} = p^+((\exists R.Self)^{\mathcal{I}})$. Furthermore, $\pi(\neg\exists R.Self) = C_{R.Self}$, and by definition $(C_{R.Self})^{\mathcal{I}'} = p^-((\exists R.Self)^{\mathcal{I}})$. If $C \in N_o$, then $\pi(o) = o$, and by definition $o^{\mathcal{I}'} = p^+(o^{\mathcal{I}})$. Furthermore, $\pi(\neg o) = C_o$, and by definition $(C_o)^{\mathcal{I}'} = p^-(o^{\mathcal{I}})$.

For the induction, we consider selected cases (the others are similar).

1. $p^+((\mathbf{D} \sqcup \mathbf{E})^{\mathcal{I}}) = p^+(D) \cup p^+(E) = \pi(D)^{\mathcal{I}'} \cup \pi(E)^{\mathcal{I}'} = (\pi(D) \sqcup \pi(E))^{\mathcal{I}'} = \pi(D \sqcup E)^{\mathcal{I}'}$.
   $p^-((\mathbf{D} \sqcup \mathbf{E})^{\mathcal{I}}) = p^-(D) \cap p^-(E) = \pi(\neg D)^{\mathcal{I}'} \cap \pi(\neg E)^{\mathcal{I}'} = (\pi(\neg D) \sqcap \pi(\neg E))^{\mathcal{I}'} = (\pi(\neg(D \sqcup E))^{\mathcal{I}'}$.
2. $p^+((\exists\mathbf{R}.\mathbf{D})^{\mathcal{I}}) = \{x|(\exists y)[(x,y) \in p^+(R^{\mathcal{I}}) \wedge y \in p^+(D^{\mathcal{I}})]\} = \{x|(\exists y)[(x,y) \in R^{\mathcal{I}'} \wedge y \in (\pi(D))^{\mathcal{I}'}]\} = (\exists R.\pi(D))^{\mathcal{I}'} = (\pi(\exists R.D))^{\mathcal{I}'}$.
   $p^-((\exists\mathbf{R}.\mathbf{D})^{\mathcal{I}}) = \{x|(\forall y)[(x,y) \in p^+(R^{\mathcal{I}}) \mapsto y \in p^-(D^{\mathcal{I}})]\} = \{x|(\forall y)[(x,y) \in R^{\mathcal{I}'} \mapsto y \in (\pi(\neg D))^{\mathcal{I}'}]\} = (\forall R.\pi(\neg D))^{\mathcal{I}'} = (\pi(\neg\exists R.D))^{\mathcal{I}'}$.
3. $p^+((\leq \mathbf{nR}.\mathbf{D})^{\mathcal{I}}) = \{x|\sharp\{y|(x,y) \in p^+(R^{\mathcal{I}}) \wedge y \notin p^-(D^{\mathcal{I}})\} \leq n\} = \{x|\sharp\{y|(x,y) \in R^{\mathcal{I}'} \wedge y \notin \pi(\neg D)^{\mathcal{I}'}\} \leq n\} = \{x|\sharp\{y|(x,y) \in R^{\mathcal{I}'} \wedge y \in (\Delta - \pi(\neg D)^{\mathcal{I}'})\} \leq n\} = \{x|\sharp\{y|(x,y) \in R^{\mathcal{I}'} \wedge y \in (\neg\pi(\neg D))^{\mathcal{I}'}\} \leq n\} = (\leq nR.\neg\pi(\neg D))^{\mathcal{I}'} = (\pi(\leq nR.D))^{\mathcal{I}'}$.
   $p^-((\leq \mathbf{nR}.\mathbf{D})^{\mathcal{I}}) = \{x|\sharp\{y|(x,y) \in p^+(R^{\mathcal{I}}) \wedge y \in p^+(D^{\mathcal{I}})\} \geq (n+1)\} = \{x|\sharp\{y|(x,y) \in R^{\mathcal{I}'} \wedge y \in \pi(D)^{\mathcal{I}'}\} \geq (n+1)\} = (\geq (n+1)R.\pi(D))^{\mathcal{I}'} = (\pi(\neg \leq nR.D))^{\mathcal{I}'}$.
   The remaining cases, involving negation, all have the same form:
4. $p^+((\neg D)^{\mathcal{I}}) = p^-(D^{\mathcal{I}}) = \pi((\neg D))^{\mathcal{I}'}$.
   $p^-((\neg D)^{\mathcal{I}}) = p^+(D^{\mathcal{I}}) = \pi(D)^{\mathcal{I}'} = \pi(\neg\neg D)^{\mathcal{I}'}$.
□

The below two lemmas are needed for the proof of Proposition 38.

Lemma 63. *For any 4-interpretation $\mathcal{I}$ and primed counterpart $\mathcal{I}'$, if S is the inverse of a role name R, then $p^+(S^{\mathcal{I}}) = S^{\mathcal{I}'}$,*

Proof. If $(a,b) \in p^+(S^{\mathcal{I}})$, then $(b,a) \in p^+(R^{\mathcal{I}})$. As such, $(b,a) \in R^{\mathcal{I}'}$, and so $(a,b) \in S^{\mathcal{I}'}$. If $(a,b) \in S^{\mathcal{I}'}$, then $(b,a) \in R^{\mathcal{I}'}$, and so both $(b,a) \in p^+(R^{\mathcal{I}})$ and $(a,b) \in p^+(S^{\mathcal{I}})$. □

Lemma 64. *For any 4-interpretation $\mathcal{I}$ and primed counterpart $\mathcal{I}'$,*

$$p^+((R_1 \circ \ldots \circ R_n)^{\mathcal{I}}) = (R_1 \circ \ldots \circ R_n)^{\mathcal{I}'}.$$

Proof. **(LR)** If $(x,z) \in p^+((R_1 \circ \ldots \circ R_n)^{\mathcal{I}})$, then there are $y_1, \ldots, y_{n-1}$ such that $(x,y_1) \in p^+(R_1^{\mathcal{I}})$, $(y_1,y_2) \in p^+(R_2^{\mathcal{I}})$, …, $(y_{n-1},z) \in p^+(R_n^{\mathcal{I}})$. Observe that for each R_i, $p^+(R_i^{\mathcal{I}}) = R_i^{\mathcal{I}'}$ by definition of $\mathcal{I}'$. And so $(x,y_1) \in R_1^{\mathcal{I}'}$, $(y_1,y_2) \in R_2^{\mathcal{I}'}$, …, $(y_{n-1},z) \in R_n^{\mathcal{I}'}$. As such, $(x,z) \in (R_1 \circ$

$\ldots \circ R_n)^{\mathcal{I}'}$. **(RL)** Suppose $(x,z) \in (R_1 \circ \ldots \circ R_n)^{\mathcal{I}'}$. Then there exist $y_1, \ldots, y_{n-1}$ such that $(x, y_1) \in R_1^{\mathcal{I}'}$, $(y_1, y_2) \in R_2^{\mathcal{I}'}$, ..., $(y_{n-1}, z) \in R_n^{\mathcal{I}'}$. For each R_i, $p^+(R_i^{\mathcal{I}}) = R_i^{\mathcal{I}'}$ by definition of $\mathcal{I}'$. And so $(x, y_1) \in p^+(R_1^{\mathcal{I}})$, $(y_1, y_2) \in p^+(R_2^{\mathcal{I}})$, ..., $(y_{n-1}, z) \in p^+(R_n^{\mathcal{I}})$. As such, $(x, z) \in p^+((R_1 \circ \ldots \circ R_n)^{\mathcal{I}})$. □

Proposition 38. For any 4-interpretation $\mathcal{I}$, $\mathcal{I}$ is a 4-model of $\mathcal{SROIQ}4$ axiom A *iff* its primed counterpart $\mathcal{I}'$ is a 2-model of $\pi(A)$.

Proof. (By cases):

1. $\mathbf{C(a)}$: $\pi(C(a)) = \pi(C)(a)$. By Prop. 37, $a^{\mathcal{I}} \in p^+(C^{\mathcal{I}})$ *iff* $a^{\mathcal{I}'} \in \pi(C)^{\mathcal{I}'}$.
2. $\mathbf{A = R(a,b)}$. $\pi(R(a,b)) = R(a,b)$. Since $p^+(R^{\mathcal{I}}) = R^{\mathcal{I}'}$ and $(a^{\mathcal{I}}, b^{\mathcal{I}}) = (a^{\mathcal{I}'}, b^{\mathcal{I}'})$, $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in p^+(R^{\mathcal{I}})$ *iff* $(a^{\mathcal{I}'}, b^{\mathcal{I}'}) \in R^{\mathcal{I}'}$.
3. $\mathbf{A = \neg R(a,b)}$. $\pi(\neg R(a,b)) = R'(a,b)$. Since $p^-(R^{\mathcal{I}}) = R'^{\mathcal{I}'}$ and $(a^{\mathcal{I}}, b^{\mathcal{I}}) = (a^{\mathcal{I}'}, b^{\mathcal{I}'})$, $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in p^+((\neg R)^{\mathcal{I}})$ *iff* $(a^{\mathcal{I}'}, b^{\mathcal{I}'}) \in R'^{\mathcal{I}'}$.
4. $\mathbf{A = (C \sqsubset D)}$:
   $p^+(C^{\mathcal{I}}) \subseteq p^+(D^{\mathcal{I}})$ *iff* $\pi(C)^{\mathcal{I}'} \subseteq \pi(D)^{\mathcal{I}'}$ *iff* $\mathcal{I}'$ 2-satisfies $(\pi(C) \sqsubset \pi(D))$ *iff* $\mathcal{I}'$ 2-satisfies $\pi(C \sqsubset D)$.
5. $\mathbf{A = (C \to D)}$: $(p^+(C^{\mathcal{I}}) \subseteq p^+(D^{\mathcal{I}})$ and $p^-(D^{\mathcal{I}}) \subseteq p^-(C^{\mathcal{I}}))$ *iff* $(\pi(C)^{\mathcal{I}'} \subseteq \pi(D)^{\mathcal{I}'}$ and $\pi(\neg D)^{\mathcal{I}'} \subseteq \pi(\neg C)^{\mathcal{I}'})$ *iff* ($\mathcal{I}'$ 2-satisfies $\{\pi(C) \sqsubseteq \pi(D),\ \pi(\neg D) \sqsubseteq \pi(\neg C)\}$ *iff* $\mathcal{I}'$ 2-satisfies $\pi(C \to D)$.
6. $\mathbf{A = (C \mapsto D)}$: $(\Delta - p^-(C^{\mathcal{I}})) \subseteq p^+(D^{\mathcal{I}})$ *iff* $(\Delta - \pi(\neg C)^{\mathcal{I}'}) \subseteq \pi(D)^{\mathcal{I}'}$ *iff* $(\neg\pi(\neg C))^{\mathcal{I}'} \subseteq \pi(D)^{\mathcal{I}'}$ *iff* $\mathcal{I}'$ 2-satisfies $(\neg\pi(\neg C) \sqsubseteq \pi(D))$ *iff* $\mathcal{I}'$ 2-satisfies $\pi(C \mapsto D)$.
7. $\mathbf{R_1 \circ \ldots \circ R_n \sqsubset R_{n+1}}$: $\mathcal{I}$ 4-satisfies $R_1 \circ \ldots \circ R_n \sqsubset R_{n+1}$ *iff*
   $p^+((R_1 \circ \ldots \circ R_n)^{\mathcal{I}}) \subseteq p^+(R_{n+1}^{\mathcal{I}})$ *iff* $(R_1 \circ \ldots \circ R_n)^{\mathcal{I}'} \subseteq R_{n+1}^{\mathcal{I}'}$ *iff* $\mathcal{I}'$ 2-satisfies $R_1 \circ \ldots \circ R_n \sqsubseteq R_{n+1}$ *iff* $\mathcal{I}'$ 2-satisfies $\pi(R_1 \circ \ldots \circ R_n \sqsubset R_{n+1})$.
8. $\mathbf{A = Ref(R)}$:$\mathcal{I}$ 4-satisfies $Ref(R)$ *iff*
   $\{(x,x) | x \in \Delta\} \subseteq p^+(R^{\mathcal{I}})$ *iff*
   $\{(x,x) | x \in \Delta\} \subseteq R^{\mathcal{I}'}$ *iff* $\mathcal{I}'$ 2-satisfies $Ref(R)$ *iff* $\mathcal{I}'$ 2-satisfies $\pi(Ref(R))$.
9. $\mathbf{A = Irr(R)}$: $\mathcal{I}$ 4-satisfies $Irr(R)$ *iff*
   $\{(x,x) | x \in \Delta\} \subseteq p^-(R^{\mathcal{I}})$ *iff*
   $\{(x,x) | x \in \Delta\} \subseteq R'^{\mathcal{I}'}$ *iff* $\mathcal{I}'$ 2-satisfies $Ref(R')$ *iff* $\mathcal{I}'$ 2-satisfies $\pi(Irr(R))$.
10. $\mathbf{A = Dis(R,S)}$: $\mathcal{I}$ 4-satisfies $Dis(R,S)$ *iff* $(p^+(R^{\mathcal{I}}) \subseteq p^-(S^{\mathcal{I}})$ and $p^+(S^{\mathcal{I}}) \subseteq p^-(R^{\mathcal{I}}))$ *iff* $(R^{\mathcal{I}'} \subseteq S'^{\mathcal{I}'}$ and $S^{\mathcal{I}'} \subseteq R'^{\mathcal{I}'})$ *iff* $\mathcal{I}'$ 2-satisfies $R \sqsubset S'$ and $S \sqsubset R'$ *iff* $\mathcal{I}'$ 2-satisfies $\pi(Dis(R,S))$.

□

Proposition 39 Let KB be a $\mathcal{SROIQ}4$ knowledge base. For any $\mathcal{SROIQ}4$ axiom A:

$$KB \models_{\mathcal{SROIQ}4} A \textit{ iff } \pi(KB) \models_{\mathcal{SROIQ}} \pi(A).$$

Proof. **(LR)** Suppose $KB \models_{\mathcal{SROIQ}4} A$ and let $\mathcal{I}'$ 2-satisfy $\pi(KB)$. Assume *wlog* that $\mathcal{I}'$ is the primed-counterpart of 4-interpretation $\mathcal{I}$ of KB. Since $\mathcal{I}'$ 2-satisfies $\pi(KB)$, by Prop. 38 $\mathcal{I}$ 4-satisfies KB and hence A. By Prop. 38, $\mathcal{I}'$ 2-satisfies $\pi(A)$. Generalizing on $\mathcal{I}'$, it follows that $\pi(KB) \models_{\mathcal{SROIQ}} \pi(A)$. **(RL)** Suppose $\pi(KB) \models_{\mathcal{SROIQ}} \pi(A)$ and that $\mathcal{I}$ 4-satisfies KB. Then there is a 2-interpretation $\mathcal{I}'$ that is the primed-counterpart of $\mathcal{I}$. By Prop. 38, since $\mathcal{I}$ 4-satisfies KB, $\mathcal{I}'$ 2-satisfies $\pi(KB)$ and so $\pi(A)$. By Prop 38, $\mathcal{I}$ 4-satisfies A. Generalizing on $\mathcal{I}$, $KB \models_{\mathcal{SROIQ}4} A$. □

A.5. Tractable DLs

Proof of Proposition 49. Let $KB \cup \{A\}$ be a set of Horn-$\mathcal{SHOIQ}\circ$ assertions, with all GCIs being $\sqsubset$-axioms. If $\pi_{\text{Horn}}(\pi(KB)) \models_2 \pi_{\text{Horn}}(\pi(A))$, then $\pi(KB) \models_2 \pi(A)$.

Proof. Suppose $\pi_{\text{Horn}}(\pi(KB)) \models_2 \pi_{\text{Horn}}(\pi(A))$ and let $\mathcal{I}$ be a model of $\pi(KB)$. Since $B^=$ does not appear in $\pi(KB)$, we may define a new interpretation $\mathcal{I}'$ by extending $\mathcal{I}$ with $B^{=\mathcal{I}'} = (\neg B')^{\mathcal{I}}$. Clearly, $\mathcal{I}'$ models every axiom of the form $B^= \sqcap B' \sqsubseteq \bot$. Furthermore, since $B^{=\mathcal{I}'} = (\neg B')^{\mathcal{I}}$, the interpretations of each concept description in $\pi(KB)$ and corresponding description in $\pi_{\text{Horn}}(\pi(KB))$ is the same, and so $\mathcal{I}'$ 2-satisfies $\pi_{\text{Horn}}(\pi(KB))$. Since $\pi_{\text{Horn}}(\pi(KB)) \models_2 \pi_{\text{Horn}}(\pi(A))$, $\mathcal{I}'$ 2-satisfies $\pi_{\text{Horn}}(\pi(A))$. However, since the interpretations of each concept description in $\pi(A)$ and corresponding description in $\pi_{\text{Horn}}(\pi(A))$ are the same, $\mathcal{I}$ 2-satisfies $\pi(A)$. □

Proof of Proposition 50. Every DL-Lite$_{core}$ and DL-Lite$_{\mathcal{R}}$ knowledge base is 4-satisfiable.

Proof. Let $\Delta^{\mathcal{I}} = \{d\}$. Let $C^{\mathcal{I}} = \langle\{d\}, \{d\}\rangle$ for each $C \in N_C$, and let $a^{\mathcal{I}} = d$ for each $a \in N_I$. For each role R, let $R^{\mathcal{I}} = \langle\{(d,d)\}, \{(d,d)\}\rangle$. Observe that for each $C \in N_C$, $C^{\mathcal{I}} = (\neg C)^{\mathcal{I}}$, and for each role R, $(\neg R)^{\mathcal{I}} = R^{\mathcal{I}}$, and $(\exists R)^{\mathcal{I}} = (\neg\exists R)^{\mathcal{I}} = \langle\{d\},\{d\}\rangle$. It is clear that each GCI, RIA, class assertion, and role assertion is 4-satisfied. □

For the remaining proofs about DL-Lite, we abbreviate π_{Lite} to π. Let $\mathcal{I}$ be a four-valued interpretation. In order to accommodate the modified paraconsistent semantics for DL-Lite and its translation, we extend the definition of the primed counterpart $\mathcal{I}'$ of $\mathcal{I}$: For each role $R \in N_R \cup N_R^-$, $R^{=\mathcal{I}'} =_{def} \Delta - p^-(R^{\mathcal{I}})$.

Lemma 65. *For any 4-interpretation $\mathcal{I}$, primed counterpart $\mathcal{I}'$, and DL-Lite concept C, $p^+((C)^{\mathcal{I}}) = \pi(C)^{\mathcal{I}'}$, and $p^-((C)^{\mathcal{I}}) = \pi(\neg C)^{\mathcal{I}'}$.*

Proof. The cases for atomic concepts A and their negations $\neg A$ proceed as for $\mathcal{SROIQ}4$ (see Proposition 37). We need only consider $\exists R$ and $\neg\exists R$.

1. If $C = \exists R$, then $\pi(\exists R) = \exists R$.
   (a) $p^+((\exists R)^{\mathcal{I}}) = \{x|(\exists y)[(x,y) \in p^+(R^{\mathcal{I}})]\}$. However, $p^+(R^{\mathcal{I}}) = R^{\mathcal{I}'}$, and so $p^+((\exists A)^{\mathcal{I}}) = (\exists R)^{\mathcal{I}'} = \pi(\exists R)^{\mathcal{I}'}$.
   (b) $p^-((\exists R)^{\mathcal{I}}) = \{x|(\forall y)[(x,y) \in p^-(R^{\mathcal{I}})]\}$. However, $R^{=\mathcal{I}'} = \Delta - p^-(R^{\mathcal{I}})$, and so $p^-((\exists R)^{\mathcal{I}}) = \{x|(\forall y)[(x,y) \notin R^{=\mathcal{I}'}]\}$. So, $p^-((\exists R)^{\mathcal{I}}) = (\neg\exists R^=)^{\mathcal{I}'} = \pi(\neg\exists R)^{\mathcal{I}'}$.
2. If $C = \neg\exists R$, then $\pi(\neg\exists R) = \neg\exists R^=$.
   (a) $p^+((\neg\exists R)^{\mathcal{I}}) = p^-((\exists R)^{\mathcal{I}}) = \{x|(\forall y)[(x,y) \in p^-(R^{\mathcal{I}})]\}$. Since $R^{=\mathcal{I}'} = \Delta - p^-(R^{\mathcal{I}})$, it follows that $p^+((\neg\exists R)^{\mathcal{I}}) = \{x|(\forall y)[(x,y) \notin R^{=\mathcal{I}'}]\}$. And so $p^+((\neg\exists R)^{\mathcal{I}}) = (\neg\exists R^=)^{\mathcal{I}'}$.
   (b) $p^-((\neg\exists R)^{\mathcal{I}}) = \{x|(\exists y)[(x,y) \in p^+(R^{\mathcal{I}})]\}$. However, $p^+(R^{\mathcal{I}}) = R^{\mathcal{I}'}$, so $p^-((\neg\exists R)^{\mathcal{I}}) = (\exists R)^{\mathcal{I}'} = \pi(\exists R)^{\mathcal{I}'} = \pi(\neg\neg\exists R)^{\mathcal{I}'}$.

□

Lemma 66. *Let $\mathcal{I}$ be a 4-interpretation and $\mathcal{I}'$ its 2-counterpart. $\mathcal{I}$ is a 4-model of DL-Lite axiom A* iff *$\mathcal{I}'$ is a 2-model of $\pi(X)$.*

Proof. For axioms of the form $C(a)$, $R(a,b)$, $C \sqsubset D$, $C \to D$, $C \mapsto D$, the proof for Proposition 38 can be used (with Lemma 65 substituted appropriately). The cases for RIAs are given below.

1. $\mathbf{A} = (\mathbf{R} \sqsubset \mathbf{S})$: $\mathcal{I}$ 4-satisfies $R \sqsubset S$ *iff* $p^+(R^{\mathcal{I}}) \subseteq p^+(S^{\mathcal{I}})$ *iff* $R^{\mathcal{I}'} \subseteq S^{\mathcal{I}'}$ *iff* $\mathcal{I}'$ 2-satisfies $R \sqsubseteq S$ *iff* $\mathcal{I}'$ 2-satisfies $\pi(R \sqsubset S)$.
2. $\mathbf{A} = (\mathbf{R} \sqsubset \neg\mathbf{S})$: $\mathcal{I}$ 4-satisfies $R \sqsubset \neg S$ *iff* $p^+(R^{\mathcal{I}}) \subseteq p^-(S^{\mathcal{I}})$ *iff* $R^{\mathcal{I}'} \subseteq S'^{\mathcal{I}'}$ *iff* $\mathcal{I}'$ 2-satisfies $R \sqsubseteq S'$ *iff* $\mathcal{I}'$ 2-satisfies $\pi(R \sqsubset \neg S)$.

□

Proof of Proposition 52. For any DL-Lite ontology KB,

$$KB \models_4 A \textit{ iff } \pi_{\text{Lite}}(KB) \models_{\text{DL-Lite}_{\mathcal{R}}} \pi_{\text{Lite}}(A).$$

Proof. **(LR)** Suppose $KB \models_4 A$ and let $\mathcal{I}_1$ be a 2-model of $\pi(KB)$. For each role R, $R^{=\mathcal{I}_1} \subseteq \Delta - R'^{\mathcal{I}_1}$. Since R' can only appear on the right-hand side of an axiom of the form $S \sqsubseteq R'$, $\mathcal{I}_1$ can be extended to a 2-model $\mathcal{I}'$ of $\pi(KB)$ such that $R^{=\mathcal{I}'} = \Delta - R'^{\mathcal{I}_1}$. Since $\mathcal{I}'$ 2-satisfies $\pi(KB)$, then by Lemma 66, $\mathcal{I}$ 4-satisfies KB. And so $\mathcal{I}$ 4-satisfies A. As such, $\mathcal{I}'$ 2-satisfies $\pi(A)$. We now show $\mathcal{I}_1$ 2-satisfies $\pi(A)$, considering the form of A.

If A is $C(a)$, $R(a,b)$, GCI $C \sqsubset D$, RIA $C \sqsubset \exists R$, or RIA $R \sqsubseteq S$, where R is atomic and C, D, and S are atomic or simple negations, then since $\mathcal{I}$ and $\mathcal{I}_1$ differ only in the extension of $R^=$ and $R^=$ doesn't appear in $\pi(A)$, $\mathcal{I}_1$ 2-satisfies $\pi(A)$.

If $A = C \sqsubset \neg\exists R$, then $\pi(A) = \pi(C) \sqsubset \neg\exists R^=$, and so $\pi(C)^{\mathcal{I}'} \subseteq (\neg\exists R^=)^{\mathcal{I}'}$. Note that $\pi(C)^{\mathcal{I}'} = \pi(C)^{\mathcal{I}_1}$. Since $R^{=\mathcal{I}'} = \Delta - R'^{\mathcal{I}_1}$, $\{x|(\forall y)[(x,y) \notin R^{=\mathcal{I}'}]\}$ is a subset of $\{x|(\forall y)[(x,y) \notin R^{=\mathcal{I}_1}]\}$. From this, $(\neg\exists R^=)^{\mathcal{I}'} \subseteq (\neg\exists R^=)^{\mathcal{I}_1}$. As such, $\pi(C)^{\mathcal{I}_1} \subseteq (\neg\exists R^=)^{\mathcal{I}_1}$, and so $\mathcal{I}_1$ 2-satisfies $\pi(C \sqsubset \neg\exists R)$.

(RL) Suppose $\pi(KB) \models \pi(A)$ and $\mathcal{I}$ 4-satisfies KB. $\mathcal{I}$ corresponds to a 2-interpretation $\mathcal{I}'$ of $\pi(KB \cup \{A\})$. By Lemma 66, $\mathcal{I}'$ 2-satisfies $\pi(KB)$ and hence $\pi(A)$. By Lemma 66, $\mathcal{I}$ 4-satisfies A. □

A.6. ELP

Proof of Proposition 57. Let KB be a $\supset$-rule base and KB' the result of applying one of H1,…,H4, B1,…,B3 to KB. KB is 4-satisfiable *iff* KB' is.

Proof. H2, H3, and B2 are trivial enough to omit. We consider the other cases. Let r and r' denote the old and new rules, respectively. KB' potentially introduces new variables and atomic concepts. However, each interpretation for KB' extends one for KB, and each interpretation for KB can be extended into one for KB'.

H1: Since $body(r) = body(r')$, $\mathcal{I}, \sigma_{\mathcal{I}} \not\models_t body(r)$ *iff* $\mathcal{I}, \sigma_{\mathcal{I}} \not\models_t body(r')$. Obviously, $t^{\mathcal{I}} \in p^+((C \sqcap D)^{\mathcal{I}})$ *iff* $\mathcal{I}, \sigma_{\mathcal{I}} \models_t (C(t) \wedge D(t))$. The rest of $head(r')$ is unchanged from r, and so $\mathcal{I}, \sigma_{\mathcal{I}} \models_t head(r)$ *iff* $\mathcal{I}, \sigma_{\mathcal{I}} \models_t head(r')$. So, $\mathcal{I}, \sigma_{\mathcal{I}} \models_t r$ *iff* $\mathcal{I}, \sigma_{\mathcal{I}} \models_t r'$.

H4: Let $s = A(x) \supset C(x)$ be the axiom added. **(LR)** Suppose $\mathcal{I}, \sigma_\mathcal{I} \models_t r$, and let $\mathcal{I}'$ be obtained by extending $\mathcal{I}$ so that $A^{\mathcal{I}'} = C^{\mathcal{I}}$ (ensuring $\mathcal{I}', \sigma_{\mathcal{I}'} \models_t s$). Since $A^{\mathcal{I}'} = C^{\mathcal{I}}$, it must be that $\mathcal{I}, \sigma_\mathcal{I} \models_t r'$. **(RL)** Suppose $\mathcal{I}, \sigma_\mathcal{I} \models_t r'$ and $\mathcal{I}, \sigma_\mathcal{I} \models_t s$ for some $\mathcal{I}$. If $\mathcal{I}, \sigma_\mathcal{I} \not\models_t \mathit{body}(r')$, then $\mathcal{I}, \sigma_\mathcal{I} \models_t r$. If $\mathcal{I}, \sigma_\mathcal{I} \models_t \mathit{head}(r')$, then $\mathcal{I}, \sigma_\mathcal{I} \models_t \exists R.A(t)$, and since $\mathcal{I}, \sigma_\mathcal{I} \models_t s$, it follows that $\mathcal{I}, \sigma_\mathcal{I} \models_t \exists R.C(t)$. Since the rest of $\mathit{head}(r')$ matches $\mathit{head}(r)$, $\mathcal{I}, \sigma_\mathcal{I} \models_t \mathit{head}(r)$, and so $\mathcal{I}, \sigma_\mathcal{I} \models_t r$.

B1: Suppose $\mathcal{I}, \sigma_\mathcal{I} \models_t r$. Then either $\mathcal{I}, \sigma_\mathcal{I} \not\models_t \mathit{body}(r)$ or else $\mathcal{I}, \sigma_\mathcal{I} \models_t \mathit{head}(r)$. If $\mathcal{I}, \sigma_\mathcal{I} \models_t \mathit{head}(r)$, then $\mathcal{I}, \sigma_\mathcal{I} \models_t \mathit{head}(r')$, and so $\mathcal{I}, \sigma_\mathcal{I} \models_t r'$. If $\mathcal{I}, \sigma_\mathcal{I} \not\models_t \mathit{body}(r)$, then $\mathcal{I}, \sigma_\mathcal{I} \not\models_t A$ for some $A \in \mathit{body}(r)$. If $A \in \mathit{body}(r')$, then $\mathcal{I}, \sigma_\mathcal{I} \not\models_t \mathit{body}(r')$. If $A \notin \mathit{body}(r')$, then $A = (C \sqcap D)(t)$. As such, either $\mathcal{I}, \sigma_\mathcal{I} \not\models_t C(t)$ or else $\mathcal{I}, \sigma_\mathcal{I} \not\models_t D(t)$. Either way, $\mathcal{I}, \sigma_\mathcal{I} \not\models_t C(t) \wedge D(t)$. And so $\mathcal{I}, \sigma_\mathcal{I} \not\models_t \mathit{body}(r')$, which implies $\mathcal{I}, \sigma_\mathcal{I} \models_t r'$.
The other direction proceeds similarly.

B3: $\mathit{head}(r) = \mathit{head}(r')$, and so $\mathcal{I}, \sigma_\mathcal{I} \models_t \mathit{head}(r)$ *iff* $\mathcal{I}, \sigma_\mathcal{I} \models_t \mathit{head}(r')$. We consider the bodies: **(LR)** Suppose $\mathcal{I}, \sigma_\mathcal{I} \not\models_t \mathit{body}(r)$. If this is due to an unrelated atom (not $\exists R.C(t)$), then $\mathcal{I}, \sigma_\mathcal{I} \not\models_t \mathit{body}(r')$. So suppose $\mathcal{I}, \sigma_\mathcal{I} \not\models_t \exists R.C(t)$. By definition, $t^\mathcal{I} \notin \{z | (\exists y)[(z, y) \in p^+(R^\mathcal{I})$ and $y \in p^+(C^\mathcal{I})]\}$. So for all $d \in \Delta^\mathcal{I}$, either $(t^\mathcal{I}, d) \notin p^+(R^\mathcal{I})$ or else $d \notin p^+(C^\mathcal{I})$. In other words, for all d, either $\mathcal{I}, \sigma_\mathcal{I}[x/d] \not\models_t C(x)$ or else $\mathcal{I}, \sigma_\mathcal{I}[x/d] \not\models_t R(t, x)$. Either way, $\mathcal{I}, \sigma_\mathcal{I} \not\models_t \mathit{body}(r')$. **(RL)** Now suppose $\mathcal{I}, \sigma_\mathcal{I} \not\models_t \mathit{body}(r')$, and assume that for all $d \in \Delta^\mathcal{I}$, either $\mathcal{I}, \sigma_\mathcal{I}[x/d] \not\models_t C(x)$ or else $\mathcal{I}, \sigma_\mathcal{I}[x/d] \not\models_t R(t, x)$. Either way, $\mathcal{I}, \sigma_\mathcal{I} \not\models_t \exists R.C$. And so $\mathcal{I}, \sigma_\mathcal{I} \not\models_t \mathit{body}(r)$.

□

Proof of Proposition 58. If $\mathcal{P}$ is a well-formed formula of ELP, $\mathcal{I}$ is a 4-interpretation of $\mathcal{P}$, and $\mathcal{I}'$ is the primed counterpart of $\mathcal{I}$, then for any assignment $\sigma_\mathcal{I}$, $\mathcal{I}, \sigma_\mathcal{I} \models_t \mathcal{P}$ *iff* $\mathcal{I}, \sigma_{\mathcal{I}'} \models \pi(\mathcal{P})$.

Proof. Let $\mathcal{I}$ be a 4-interpretation and $\mathcal{I}'$ its primed counterpart. We induct on the degree of $\mathcal{P}$, considering selected cases. The rest are similar.

1. $\mathbf{C(t)}$: $\mathcal{I}, \sigma_\mathcal{I} \models_t C(t)$ *iff* $\sigma_\mathcal{I}(t) \in p^+(C^\mathcal{I})$ *iff* $\sigma_\mathcal{I}(t) \in \pi(C)^{\mathcal{I}'}$ *iff* $\mathcal{I}', \sigma_\mathcal{I} \models \pi(C)(t)$.
2. $\neg\mathbf{R(t_1, t_2)}$: $\mathcal{I}, \sigma_\mathcal{I} \models_t \neg R(t_1, t_2)$ *iff* $(\sigma_\mathcal{I}(t_1), \sigma_\mathcal{I}(t_2)) \in p^-(R^\mathcal{I})$ *iff* $(\sigma_\mathcal{I}(t_1), \sigma_\mathcal{I}(t_2)) \in R'^{\mathcal{I}}$ *iff* $(\sigma_\mathcal{I}(t_1), \sigma_\mathcal{I}(t_2)) \in \pi(\neg R)^{\mathcal{I}'}$ *iff* $\mathcal{I}', \sigma_\mathcal{I} \models \pi(\neg R)(t_1, t_2)$.
3. $(\mathcal{P} \wedge \mathcal{Q})$: $\mathcal{I}, \sigma_\mathcal{I} \models_t (\mathcal{P} \wedge \mathcal{Q})$ *iff* $(\mathcal{I}, \sigma_\mathcal{I} \models_t \mathcal{P}$ and $\mathcal{I}, \sigma_\mathcal{I} \models_t \mathcal{Q})$ *iff* $(\mathcal{I}', \sigma_\mathcal{I} \models \pi(\mathcal{P})$ and $\mathcal{I}, \sigma_{\mathcal{I}'} \models_t \pi(\mathcal{Q}))$ *iff* $\mathcal{I}', \sigma_\mathcal{I} \models \pi(\mathcal{P} \wedge \mathcal{Q})$.
4. $(\mathcal{P} \supset \mathcal{Q})$: $\mathcal{I}, \sigma_\mathcal{I} \models_t (\mathcal{P} \supset \mathcal{Q})$ *iff* $(\mathcal{I}, \sigma_\mathcal{I} \not\models_t \mathcal{P}$ or $\mathcal{I}, \sigma_\mathcal{I} \models_t \mathcal{Q})$ *iff* $(\mathcal{I}', \sigma_\mathcal{I} \not\models \pi(\mathcal{P})$ or $\mathcal{I}, \sigma_{\mathcal{I}'} \models_t \pi(\mathcal{Q}))$ *iff* $(\mathcal{I}', \sigma_\mathcal{I} \models \pi(\mathcal{P}) \mapsto \pi(\mathcal{Q}))$ *iff* $\mathcal{I}', \sigma_\mathcal{I} \models \pi(\mathcal{P} \supset \mathcal{Q})$.
5. $(\forall \mathbf{x})\mathcal{P}$, x not safe: $\mathcal{I}, \sigma_\mathcal{I} \models_t (\forall x)\mathcal{P}$ *iff* $\mathcal{I}, \sigma_\mathcal{I}[x/d] \models_t \mathcal{P}$ for all $d \in \Delta$ *iff* $\mathcal{I}', \sigma_\mathcal{I}[x/d] \models \mathcal{P}$ for all $d \in \Delta$ *iff* $\mathcal{I}', \sigma_\mathcal{I} \models (\forall x)\mathcal{P}$.
6. $(\forall \mathbf{x})\mathcal{P}$, x safe: $\mathcal{I}, \sigma_\mathcal{I} \models_t (\forall x)\mathcal{P}$ *iff* $\mathcal{I}, \sigma_\mathcal{I}[x/d] \models_t \mathcal{P}$ for all named $d \in \Delta$ *iff* $\mathcal{I}', \sigma_\mathcal{I}[x/d] \models \mathcal{P}$ for all named $d \in \Delta$ *iff* $\mathcal{I}', \sigma_\mathcal{I} \models (\forall x)\mathcal{P}$.

□

Proof of Proposition 59. Let KB be an ELP rule base and $\mathcal{P}$ a well-formed formula of ELP. $KB \models_{ELP4} \mathcal{P}$ *iff* $\pi(KB) \models_{ELP} \pi(\mathcal{P})$.

Proof. **(LR)** Suppose $KB \models_{ELP4} \mathcal{P}$ and let $\mathcal{I}'$ be a model of $\pi(KB)$ and $\sigma_{\mathcal{I}'}$ an assignment for $\mathcal{I}'$. Assume *wlog* that $\mathcal{I}'$ is the primed-counterpart of a 4-interpretation $\mathcal{I}$. Since $\mathcal{I}'$ models $\pi(KB)$, $\mathcal{I}', \sigma_{\mathcal{I}'} \models_{ELP} \pi(KB)$. From Prop. 58, $\mathcal{I}, \sigma_\mathcal{I} \models_{ELP4} KB$ and so $\mathcal{I}, \sigma_\mathcal{I} \models_{ELP4} \mathcal{P}$. From Prop. 58, $\mathcal{I}', \sigma_{\mathcal{I}'} \models_{ELP} \pi(\mathcal{P})$. **(RL)** Suppose $\pi(KB) \models_{ELP} \pi(\mathcal{P})$ and let $\mathcal{I}$ 4-model KB and let $\sigma_\mathcal{I}$ be an assignment for $\mathcal{I}$. Let $\mathcal{I}'$ be the primed-counterpart of $\mathcal{I}$. Since $\mathcal{I}$ 4-satisfies KB, $\mathcal{I}, \sigma_\mathcal{I} \models_{ELP4} KB$. From Prop. 58, $\mathcal{I}', \sigma_{\mathcal{I}'} \models_{ELP} \pi(KB)$ and so $\mathcal{I}', \sigma_{\mathcal{I}'} \models_{ELP} \pi(\mathcal{P})$. From Prop. 58, $\mathcal{I}, \sigma_\mathcal{I} \models_{ELP4} \mathcal{P}$. □

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